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Analysis and control of fluid–structure systems with boundary conditions involving the pressure

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À ma famille,

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List of publications

The following is a list of publications/unpublished manuscripts resulting from this thesis.

1. J. J. Casanova. Fluid structure system with boundary conditions involving the pressure. Submitted.

Declaration

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

This thesis includes one unpublished manuscript. The core theme of the thesis is mathematical analysis. The ideas, development and writing up of the paper in the thesis were the principal responsibility of myself, the candidate, working within the Institut de Mathématiques de Toulouse, University Toulouse Paul Sabatier, and School of Mathematical Sciences, Monash University, under the joint supervision of Prof. Jean-Pierre Raymond and A/Prof. Jérôme Droniou.

The submitted manuscript in the list above corresponds to Chapter 2 in this thesis.

Student signature: [redacted] **Date:** May 5, 2018

The undersigned hereby certify that the above declaration correctly reflects the nature and extent of the student and co-authors' contributions to this work.

Supervisor signature: [redacted] **Date:** May 5, 2018

Supervisor signature: [redacted] **Date:** May 5, 2018

Chapter 1

Introduction

1.1 Context

In this thesis we study mathematical models coupling a fluid and a deformable elastic structure. Such models are used to describe many physical phenomena such that the air flow around aircraft wings or the blood flow in arteries. In these situations the fluid applies a force on the structure and, in return, the displacement of the structure impacts the fluid. The interactions between the fluid and the structure are driven by this action–reaction principle and interface conditions.

The fluid–structure models considered throughout this thesis are system coupling the incompressible Navier–Stokes equations in a 2D rectangular domain with a damped Euler–Bernoulli beam equation, where the beam is a part of the upper boundary of the domain occupied by the fluid. Motivated by the study of blood flow through human arteries we prescribe pressure on the inflow and the outflow boundaries. Typically, one can think about a fluid in a vessel where the value of the pressure at both ends is fixed.

Our study starts with the existence of strong solutions for this system. In Chapter 2 we prove the local-in-time existence of strong solutions without any smallness assumption on the initial displacement of the structure. This is done using a change of variables adapted to this deformation.

We then consider in Chapter 3 the existence of time periodic solutions for this system. These solutions correspond to the normal behaviour of arteries with the periodic impulse prescribed by the heartbeat. Using regularisation properties of parabolic equations and the time periodicity of the problem we are able to construct classical solutions that are periodic in time.

Finally we study in Chapter 4 the stabilization of the previous system in a neighbourhood of a periodic solution, with a control acting on a part of the fluid boundary. We construct a parabolic evolution operator for the underlying linear system and we prove, using Floquet theory, that the linear system can be stabilized.

1.2 Presentation of the results

In the following, spaces written in boldface correspond to the vector version (in dimension 2) of the standard space. So, for example, $\mathbf{L}^2(X) = (L^2(X))^2$.

1.2.1 Chapter 2: Fluid–structure system with boundary conditions involving the pressure

Background

In this chapter we consider a fluid–structure system involving the incompressible Navier–Stokes equations in a 2D rectangular type domain where the upper part of the fluid domain is a moving structure satisfying a damped Euler–Bernoulli beam equation (see Figure 1.1). Let $T > 0$, $L > 0$ and consider the spatial domain Ω in $\mathbb{R}^2$ defined by $\Omega = (0, L) \times (0, 1)$ with the boundaries $\Gamma_i = \{0\} \times (0, 1)$, $\Gamma_o = \{L\} \times (0, 1)$, $\Gamma_s = (0, L) \times \{1\}$, $\Gamma_b = (0, L) \times \{0\}$ and the notation $\Gamma_{i,o} = \Gamma_i \cup \Gamma_o$. The domain of the fluid $\Omega_{\eta(t)}$ depends on the displacement of the beam $\eta : \Gamma_s \times (0, T) \to (-1, +\infty)$,

$$\Omega_{\eta(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ 0 < y < 1 + \eta(x, 1, t)\},$$
$$\Gamma_{\eta(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ y = 1 + \eta(x, 1, t)\}.$$

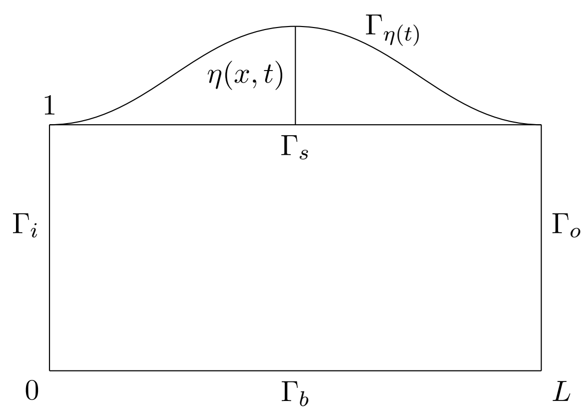

Figure 1.1: Domain $\Omega_{\eta(t)}$

The fluid equations are written in the space-time domains

$$\Sigma_T^\eta = \bigcup_{t\in(0,T)} \Gamma_{\eta(t)} \times \{t\}, \quad Q_T^\eta = \bigcup_{t\in(0,T)} \Omega_{\eta(t)} \times \{t\}.$$

$$\Sigma_T^s = \Gamma_s \times (0,T), \quad \Sigma_T^i = \Gamma_i \times (0,T), \quad \Sigma_T^o = \Gamma_o \times (0,T), \quad \Sigma_T^b = \Gamma_b \times (0,T),$$

The fluid–structure system is described by the following coupled equations:

$$\text{(NS)} \begin{cases} \mathbf{u}_t - \operatorname{div}\,\sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0 \ \text{ in } Q_T^\eta, \\ \operatorname{div}\,\mathbf{u} = 0 \ \text{ in } Q_T^\eta, \\ \mathbf{u}(0) = \mathbf{u}^0 \ \text{ in } \Omega_{\eta_1^0}, \end{cases}$$

$$\text{(1.2.1)} \qquad \text{(BC)} \begin{cases} \mathbf{u}(x,y,t) = \eta_t(x,1,t)\mathbf{e}_2 \ \text{ for } (x,y,t) \in \Sigma_T^\eta, \\ u_2 = 0 \ \text{ and } \ p + \frac{1}{2}|\mathbf{u}|^2 = 0 \ \text{ on } \Sigma_T^{i,o}, \\ \mathbf{u} = 0 \ \text{ on } \Sigma_T^b, \end{cases}$$

$$\text{(EB)} \begin{cases} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = -J_{\eta(t)}\mathbf{e}_2 \cdot \sigma(\mathbf{u},p)_{|\Gamma_{\eta(t)}}\mathbf{n}_{\eta(t)} \ \text{ on } \Sigma_T^s, \\ \eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0,L\}\times(0,T), \\ \eta(0) = \eta_1^0 \ \text{ and } \ \eta_t(0) = \eta_2^0 \ \text{ in } \Gamma_s, \end{cases}$$

(NS) are the incompressible Navier–Stokes equations. Here $\mathbf{u} = (u_1, u_2)$ is the fluid velocity, p is the pressure, $\sigma(\mathbf{u},p) = -pI + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ is the Cauchy stress tensor and ν is the viscosity of the fluid (assumed to be constant).

(BC) describes the boundary conditions of the fluid. The continuity of the fluid and structure velocities at the fluid–structure interface imposes the Dirichlet boundary condition on $\Gamma_{\eta(t)}$. For simplicity, the lower boundary of the domain is assumed to be fixed. Finally we prescribe the dynamical pressure of the fluid at the inflow and outflow boundaries.

(EB) is a damped Euler–Bernoulli beam equation. The term $-J_{\eta(t)}\mathbf{e}_2 \cdot \sigma(\mathbf{u},p)_{|\Gamma_{\eta(t)}}\mathbf{n}_{\eta(t)}$, where

$$\mathbf{n}_{\eta(t)} = J_{\eta(t)}^{-1} \begin{pmatrix} -\eta_x(x,t) \\ 1 \end{pmatrix}$$

is the normal unit vector to the curve $\Gamma_{\eta(t)}$ and $J_{\eta(t)} = \sqrt{1+\eta_x^2}$, describes the force that the fluid exerts on the structure. The non-negative coefficients α, β, γ are parameters relative to the structure. Throughout this thesis we only consider the case $\gamma > 0$ and $\alpha > 0$ (see the discussion in Section 1.3.3).

The fluid and the structure are coupled at three levels:

- The kinematic condition on $\Gamma_{\eta(t)}$.
- The dynamic condition involving $\sigma(\mathbf{u},p)$ in the beam equation.
- The geometrical coupling between the domain of the fluid and the (moving) structure.

The existence of weak solutions for this system is studied in [49]. Similar fluid–structure systems with different boundary conditions on $\Gamma_{i,o}$ have been extensively studied. In particular, for homogeneous Dirichlet boundary conditions on the in/outflow, the existence of strong solutions is established in [37]. Precisely, the results in [37] consist of a local-in-time existence of strong solutions without smallness assumptions on the data, and in the existence on fixed but arbitrary time interval $[0, T]$ for small data. For periodic boundary conditions, the existence of global strong solutions without smallness assumptions on the data is proved in [28]. The proof is based on the local existence result in [37] and new a priori estimates ensuring that the beam does not touch the bottom of the domain.

The starting point of the thesis was to adapt the techniques in [37] to boundary conditions involving the pressure of the fluid. These conditions enjoy interesting symmetry properties that can be used to recover maximal regularity results.

During our initial analysis it appeared that the proof in [37] suffers some limitations. The local-in-time existence result in this reference actually requires the initial displacement η_0^1 of the beam to be small. This issue is purely nonlinear and does not depend on the boundary conditions. After a careful study we found out that this limitation was related to the reference configuration of the fluid.

The fluid equations, in Eulerian variables, are initially written in the moving domain $\Omega_{\eta(t)}$ whereas the structure equation, with Lagrangian variables, is written in the reference configuration Γ_s. The first step in the analysis is to fix the domain using a change of variables. One possible choice is to map the moving domain $\Omega_{\eta(t)}$ onto the rectangular domain $\Omega = (0, L) \times (0, 1)$, using the transformation

$$\mathcal{T}_0(t) : \left\{ \begin{array}{l} \Omega_{\eta(t)} \longrightarrow \Omega \\ (x, y) \mapsto \left(x, \frac{y}{1+\eta(x,t)}\right). \end{array} \right. \tag{1.2.2}$$

This choice is made, for example, in [56, 37]. If we suppose that the initial displacement of the beam η_1^0 is small then $\mathcal{T}_0(t)$ is 'almost' the identity at the time $t = 0$. This is not the case for an arbitrary η_1^0. For these reasons, using the rectangular domain as a reference configuration is suitable for the existence of strong solutions on $[0, T]$ with small data, but presents difficulties in the local-in-time existence result for arbitrary data.

Contributions

The main result of this chapter, Theorem 2.4.3, can be formulated as follows.

Theorem 1.2.1. Suppose that

- $\mathbf{u}^0 \in \mathbf{H}^1(\Omega_{\eta_1^0})$ with div $\mathbf{u}^0 = 0$ in $\Omega_{\eta_1^0}$, $u_2^0 = 0$ on $\Gamma_{i,o}$ and $\mathbf{u}^0 = 0$ on Γ_b.
- $\eta_1^0 \in H^3(\Gamma_s) \cap H_0^2(\Gamma_s)$ with $\inf_{x\in(0,L)}(1 + \eta_1^0(x)) > 0$.
- $\eta_2^0 \in H_0^1(\Gamma_s)$.
- $\mathbf{u}^0$ and η_2^0 satisfy the compatibility condition $\mathbf{u}^0(x, 1 + \eta_1^0(x)) = \eta_2^0(x)\mathbf{e}_2$ for $x \in (0, L)$.

Then there exists $T > 0$ such that (1.2.1) admits a unique strong solution on $[0, T]$.

To prove this result, we tackle the question of boundary conditions involving the pressure, and we solve the issue of local-in-time existence for an initial beam displacement η_1^0 of arbitrary size.

Our idea for solving this latter issue is to use a new change of variables, based on the initial displacement of the beam (as it would be the case, for example, with a parametrization by the flow). We consider the following change of variables

$$\mathcal{T}_{\eta_1^0}(t) : \left\{ \begin{array}{l} \Omega_{\eta(t)} \longrightarrow \Omega_{\eta(0)} \\ (x, y) \mapsto \left(x, \frac{1+\eta_1^0(x)}{1+\eta(x,t)} y\right). \end{array} \right.$$

With this change of variables, the nonlinear terms which, with the change of variables (1.2.2), involved η and were therefore not small in general, now involve $\tilde{\eta} = \frac{\eta-\eta_1^0}{1+\eta_1^0}$, which is small on $[0, T]$ for small T (since $\eta(\cdot, 0) = \eta_1^0$). The estimates of these nonlinear terms then involve powers of T, and the Banach fixed point argument can be used by choosing T small enough to ensure the contraction property of the underlying mapping.

Unfortunately, the new 'curved' reference configuration $\Omega_{\eta(0)}$ of the fluid complicates the analysis of the problem. To solve the linear fluid–structure system we remove the pressure with a Leray projector Π adapted to our mixed boundary conditions (see Section 1.3.2). Obtaining strong solutions to the resulting system requires to know that the projection $\Pi\mathbf{u}$ of an $\mathbf{H}^2$ velocity $\mathbf{u}$ also belongs to $\mathbf{H}^2$. On the curved domain this property was not clear at all. The Leray projector is obtained by solving elliptic (steady) equations, and recovering the $\mathbf{H}^2$-regularity for $\Pi\mathbf{u}$ requires to find H^3-solutions for these elliptic equations, which are written in $\Omega_{\eta(0)}$ which upper part is described by η_1^0, a function that belongs to $H^3(\Gamma_s)$, not $\mathcal{C}^{2,1}(\Gamma_s)$. The classical regularity results do not apply in that case. However, using a suitable transport of H^3-functions, a symmetry argument and a bootstrap procedure we were able to establish this H^3 regularity (see Lemma 2.5.4).

The second contribution of this chapter is the handling of pressure boundary conditions. These boundary conditions are introduced in [17] in a variational framework (for weak solutions) and higher regularity results are proved in [11, 12] for smooth boundaries. Here the fluid equations present a junction between a Dirichlet boundary condition and a pressure boundary condition at a corner. To recover the expected $\mathbf{H}^2$-regularity for the fluid in a neighbourhood of the corners, we start from the steady variational formulation and use a symmetry argument. This result is stated in Theorem 2.5.4 in Section 2.5. Simultaneously, we also solve some interesting lifting questions for non-homogeneous Dirichlet boundary conditions on the upper part of the domain. Finally we extend these results to the unsteady Stokes equations. For regular data we directly obtain optimal regularity results using semigroups theory. For weaker data, our result is not optimal (optimality is a difficult and open question) but provides enough regularity to handle the fluid–structure linear system, see Lemma 2.5.3.

After all these preliminaries, we use a classical procedure (linearization, estimates of the nonlinear terms, fixed point argument) to prove the existence and the uniqueness of a local-in-time strong solution for the fluid–structure system without any smallness assumptions on the initial data. Moreover, as a side result, we obtain the existence of strong solution for arbitrary time interval $[0, T]$ for small data.

1.2.2 Chapter 3: Existence of time-periodic solutions to a fluid–structure system

Background

The objective of this chapter is to prove the existence of time-periodic solutions for the following system

$$
\begin{aligned}
&\mathbf{u}_t - \operatorname{div}\, \sigma(\mathbf{u}, p) + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0, \ \ \operatorname{div}\, \mathbf{u} = 0 \ \text{ in } Q_T^\eta,\\
&\mathbf{u} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma_T^\eta,\\
&\mathbf{u} = \boldsymbol{\omega}_1 \ \text{ on } \Sigma_T^i,\\
&u_2 = 0 \ \text{ and } \ p + \frac{1}{2}|\mathbf{u}|^2 = \omega_2 \ \text{ on } \Sigma_T^o,\\
&\mathbf{u} = 0 \ \text{ on } \Sigma_T^b, \ \ \mathbf{u}(0) = \mathbf{u}(T) \ \text{ in } \Omega_{\eta(0)},\\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = -J_{\eta(t)}\mathbf{e}_2 \cdot \sigma(\mathbf{u}, p)_{|\Gamma_{\eta(t)}} \mathbf{n}_{\eta(t)} \ \text{ on } \Sigma_T^s,\\
&\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0, L\} \times (0, T)\\
&\eta(0) = \eta(T) \ \text{ and } \ \eta_t(0) = \eta_t(T) \ \text{ in } \Gamma_s.
\end{aligned}
\tag{1.2.3}
$$

Periodic behaviours play a special role in biological phenomena. The blood flow through arteries is driven by the heartbeat, which is, in normal condition, periodic. Hence one can expect a periodic response to the system coupling the vessel and the blood flow.

To our knowledge, the existence of time-periodic solutions for (1.2.3) has not been studied yet. For abstract evolution equations, existence and behavour of periodic solutions are investigated, for example, in [13] and in [18, 40, 41, 42] for time dependent operators.

Contributions

In Chapter 3 we prove the following theorem (Theorem 3.3.2), in which the $\sharp$ symbol indicates spaces of time-periodic functions.

Theorem 1.2.2. Fix $\theta \in (0,1)$ and $T > 0$. There exists $R > 0$ such that, for all T-periodic source terms

$$(\boldsymbol{\omega}_1, \omega_2) \in \left(\mathcal{C}^\theta_\sharp([0,T]; \mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; \mathbf{H}^{-1/2}(\Gamma_i)) \right) \times \mathcal{C}^\theta_\sharp([0,T]; H^{1/2}(\Gamma_o))$$

satisfying

$$\|\boldsymbol{\omega}_1\|_{\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i))\cap\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o))} \leq R,$$

the system (1.2.3) admits a T-periodic strict solution $(\mathbf{u}, p, \eta)$ belonging to (after a change of variables mapping $\Omega_{\eta(t)}$ into Ω)

- $\mathbf{u} \in \mathcal{C}^\theta_\sharp([0,T]; \mathbf{H}^2(\Omega)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; \mathbf{L}^2(\Omega))$.
- $p \in \mathcal{C}^\theta_\sharp([0,T]; H^1(\Omega))$.
- $\eta \in \mathcal{C}^\theta_\sharp([0,T]; H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; H^2_0(\Gamma_s)) \cap \mathcal{C}^{2+\theta}_\sharp([0,T]; L^2(\Gamma_s))$.

The main idea to prove the existence of periodic solutions for the fluid–structure system is to use the matrix formulation of the linear system introduced in Chapter 2. We can then develop criteria on abstract evolution equations to ensure the existence of a periodic solution for the linear system. The existence of a periodic solution for the nonlinear system is then obtained by a fixed point argument (with smallness assumptions on the periodic source terms). The fixed point procedure is as follows:

- Choose a T-periodic triplet $(\mathbf{u}, p, \eta)$.
- Compute the nonlinear terms of the system $F(\mathbf{u}, p, \eta)$.
- Take $F(\mathbf{u}, p, \eta)$ as source term in the linear system and find a periodic solution $(\mathbf{u}^*, p^*, \eta^*)$ (this consists, for each source term, in finding an appropriate initial condition that generates a periodic solution).
- Use a Banach fixed point to find a solution such that $(\mathbf{u}, p, \eta) = (\mathbf{u}^*, p^*, \eta^*)$.

In Chapter 2 we were interested in the existence of strict solutions in L^2 (see Definition 3.4.1 for the terminology). Following this work, we initially aim at obtaining strict periodic solutions in L^2 for the fluid–structure system. For an abstract periodic evolution equation

$$\begin{aligned} & y'(t) = Ay(t) + f(t),\, t \in [0, T], \\ & y(0) = y(T), \end{aligned} \tag{1.2.4}$$

the idea to prove the existence of a strict solution in L^2 is to use the Duhamel formula to write an equation on $y(0)$. A spectral criteria (satisfied by the fluid–structure system) is then used to ensure that this equation admits a unique solution. This leads to the existence of a periodic strict solution for (1.2.4) when f belongs to L^2 (in time), see Theorem 3.4.1 in the appendix of Chapter 3.

Motivated by the stabilization of the system around a periodic solution we then investigate the existence of strict solution with a Hölder regularity in time. In the continuous framework of parabolic equations, the existence of strict solutions imposes very restrictive conditions on the initial data and on the source term at time $t = 0$. However, using the periodicity of the problem and the regularization properties of analytic semigroups, we are able to prove the existence of strict periodic solutions with Hölder regularity in time.

The spectral criteria we utilize has already been established in [40] for time-dependent operators $A(t)$ and data with Hölder regularity in time. Here we recover this criteria with a simpler proof in the stationary case. In the case of periodic source terms that are only L^2 in time, however, our existence result of periodic solutions seems to be new.

Let us finally remark that the techniques developed in Chapter 3 do not depend on the boundary conditions taken at the inflow and the outflow, once the proper regularity results for the linearized system have been established. The non-homogeneous Dirichlet boundary condition that we consider on the inflow prepares the stabilization of the system but does not impact the analysis of the existence of a periodic solution.

1.2.3 Chapter 4: Stabilization of a time-periodic fluid–structure system

Background

Following the existence of a periodic solution established in the previous chapter, we are now interested in the stabilization of the fluid–structure system in a neighbourhood of a periodic solution with a control acting on the inflow boundary of the fluid.

Boundary control is studied in [44] for a fluid–structure system with periodic boundary conditions and a structure driven by a damped wave equation. The system is exponentially stabilized around the zero solution.

In the context of blood flow through arteries, the body has several tools to control the flow rate. For instance, the vessel can contract or dilate to regulate this flow rate. In our model this corresponds to a force term applied on the beam. This type of control is used in [56] for the stabilization around the zero solution of the fluid–structure system with homogeneous boundary conditions on the in/outflow. This result is based on a unique continuation property for an eigenvalue problem proved in [50, 51].

When stabilized around a non zero solution, the fluid–structure mathematical model is perturbed. For example, after linearization, Oseen equations appear in the fluid part. The unique continuation property used in [50, 51] cannot be checked in these cases, but can be verified using numerical techniques. This approach is applied in [23, 24] for a fluid–structure system in a polygonal domain with mixed boundary conditions. In our case, owing to the usage of a boundary control acting on the inflow, we can completely prove the associated unique continuation property.

To our knowledge the stabilization around a non zero time-dependent solution is new. It is also quite challenging. The perturbed solution and the time-periodic solutions evolve in different domains. The several change of variables used to compare these solutions in a fixed domain vastly complicate the system. In particular, the underlying linear system involves an unbounded operator $\mathcal{A}(t)$ with a domain depending on time. The construction of a parabolic evolution operator in this framework requires the use of advanced theory on parabolic equations developed in [4].

Contributions

The stabilization result presented in this chapter, Theorem 4.7.2, is stated for a linear system, specified below, that is at the core of the analysis for the complete non-linear system; the stabilisation analysis for this complete model is an ongoing work (see the perspectives in Chapter 5 for more details). Let $(\mathbf{u}_\pi, p_\pi, \eta_\pi)$ be a T-periodic solution to

$$
\begin{aligned}
&\mathbf{u}_{\pi,t} + (\mathbf{u}_\pi \cdot \nabla)\mathbf{u}_\pi - \operatorname{div} \sigma(\mathbf{u}_\pi, p_\pi) = 0, \ \operatorname{div} \mathbf{u}_\pi = 0 \ \text{ in } Q_T^{\eta_\pi},\\
&\mathbf{u}_\pi(x,y,t) = \eta_{\pi,t}(x,1,t)\mathbf{e}_2 \ \text{ for } (x,y,t) \in \Sigma_T^{\eta_\pi},\\
&\mathbf{u}_\pi = \boldsymbol{\omega}_1 \ \text{ on } \Sigma_T^i,\\
&u_{\pi,2} = 0 \ \text{ and } \ p_\pi = \omega_2 \ \text{ on } \Sigma_T^o,\\
&\mathbf{u}_\pi = 0 \ \text{ on } \Sigma_T^b, \ \mathbf{u}_\pi(0) = \mathbf{u}_\pi(T) \ \text{ in } \Omega_{\eta_\pi(0)},\\
&\eta_{\pi,tt} - \beta\eta_{\pi,xx} - \gamma\eta_{\pi,txx} + \alpha\eta_{\pi,xxxx} = -J_{\eta_\pi(t)}\mathbf{e}_2 \cdot \sigma(\mathbf{u}_\pi, p_\pi)_{|\Gamma_{\eta_\pi(t)}}\mathbf{n}_{\eta_\pi(t)} \ \text{ on } \Sigma_T^s,\\
&\eta_\pi = 0 \ \text{ and } \ \eta_{\pi,x} = 0 \ \text{ on } \{0,L\} \times (0,T),\\
&\eta_\pi(0) = \eta_\pi(T) \ \text{ and } \ \eta_{\pi,t}(0) = \eta_{\pi,t}(T) \ \text{ in } \Gamma_s,
\end{aligned}
\tag{1.2.5}
$$

where $(\boldsymbol{\omega}_1, \omega_2)$ are T-periodic source terms, and $(\mathbf{u}, p, \eta)$ be a perturbation of this system, i.e. a solution to

$$
\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\,\sigma(\mathbf{u},p) = 0, \;\; \operatorname{div}\,\mathbf{u} = 0 \;\text{ in } Q^\eta_\infty,\\
&\mathbf{u}(x,y,t) = \eta_t(x,1,t)\mathbf{e}_2 \;\text{ for } (x,y,t)\in\Sigma^\eta_\infty,\\
&\mathbf{u} = \boldsymbol{\omega}_1 + \mathbf{u}_c \;\text{ on } \Sigma^i_\infty,\\
&u_2 = 0 \;\text{ and }\; p = \omega_2 \;\text{ on } \Sigma^o_\infty,\\
&\mathbf{u} = 0 \;\text{ on } \Sigma^b_\infty, \;\; \mathbf{u}(0) = \mathbf{u}^0 \;\text{ in } \Omega_{\eta^0_1},\\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = -J_{\eta(t)}\mathbf{e}_2\cdot\sigma(\mathbf{u},p)_{|\Gamma_{\eta(t)}}\mathbf{n}_{\eta(t)} \;\text{ on } \Sigma^s_\infty\\
&\eta = 0 \;\text{ and }\; \eta_x = 0 \;\text{ on } \{0,L\}\times(0,\infty),\\
&\eta(0) = \eta^0_1 \;\text{ and }\; \eta_t(0) = \eta^0_2 \;\text{ in } \Gamma_s.
\end{aligned}
\tag{1.2.6}
$$

with $(\mathbf{u}^0, \eta^0_1, \eta^0_2)$ in a neighbourhood of $(\mathbf{u}_\pi(0), \eta_\pi(0), \eta_{\pi,t}(0)) =: (\mathbf{u}^0_\pi, \eta^0_{\pi,1}, \eta^0_{\pi,2})$. We want to stabilize (1.2.6) around $(\mathbf{u}_\pi, p_\pi, \eta_\pi)$ with a control $\mathbf{u}_c$ acting on Σ^i_∞. The linear system considered throughout this chapter is obtained after the following procedure (see the dedicated Section 1.3.1 for details on the different changes of variables involved):

- **Step 1:** *Map* $\Omega_{\eta(t)}$ *into* $\Omega_{\eta_\pi(t)}$.

  The solution $(\mathbf{u}, p, \eta)$ is written in $\Omega_{\eta(t)}$ whereas the periodic solution is written in $\Omega_{\eta_\pi(t)}$. To compare the two solutions we write the perturbed solution in the periodic domain.

- **Step 2:** *Linearization.*

  We then consider the system satisfied by the difference of the two solutions, denoted $(\hat{\mathbf{u}}, \hat{p}, \hat{\eta})$, and we linearize around $(\mathbf{0}, 0, 0)$.

- **Step 3:** *Map* $\Omega_{\eta_\pi(t)}$ *into* $\Omega_{\eta^0_{\pi,1}}$.

  The linear system is still written in the time-dependent domain $\Omega_{\eta_\pi(t)}$. To fix this domain, we map $\Omega_{\eta_\pi(t)}$ into $\Omega_{\eta^0_{\pi,1}}$. When doing so, the linear system is strongly perturbed. At this step we add the following assumption:

  Assumption: The periodic beam displacement η_π remains in a small 'cylinder', i.e., $\eta_\pi(t) - \eta_\pi(s)$ is small for all t, s.

  Under this assumption we remove from the linear system the linear terms involving the difference $\eta_\pi(t) - \eta_\pi(0)$, and we obtain the final linear system studied in this chapter. The idea behind this removal is that these terms involving the small coefficient $\eta_\pi(t) - \eta_\pi(0)$ will be, later on, dealt with in a similar way as the (small) non-linearities involved in (1.2.6).

Following the previous steps we obtain the system

$$
\begin{aligned}
& \mathbf{v}_t + (\overline{\mathbf{u}}_\pi \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\overline{\mathbf{u}}_\pi + C_\pi \mathbf{v}_z - \nu \Delta \mathbf{v} + \nabla q - A_{1,1}\hat{\eta}_1 - A_2 \hat{\eta}_2 = 0 \ \text{ in } Q_\infty^{\pi,0}, \\
& \text{div } \mathbf{v} = A_3 \hat{\eta}_1 \ \text{ in } Q_\infty^{\pi,0}, \\
& \mathbf{v} = \hat{\eta}_2 \mathbf{e}_2 \ \text{ on } \Sigma_\infty^{\pi,0}, \\
& \mathbf{v} = \mathbf{u}_c \ \text{ on } \Sigma_\infty^{i}, \\
& v_2 = 0 \ \text{ and } \ q = 0, \ \text{ on } \Sigma_\infty^{o}, \\
& \mathbf{v} = 0 \ \text{ on } \Sigma_\infty^{b}, \ \ \mathbf{v}(0) = \mathbf{v}^0 \ \text{ in } \Omega_{\pi,0}, \\
& \hat{\eta}_{1,t} = \hat{\eta}_2 \text{ on } \Sigma_\infty^{s}, \\
& \hat{\eta}_{2,t} - \beta \hat{\eta}_{1,xx} - \gamma \hat{\eta}_{2,xx} + \alpha \hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1 \\
& \qquad\qquad\qquad = -J_{\eta^0_{\pi,1}} \mathbf{e}_2 \cdot \sigma(\mathbf{v}, q)_{|\Gamma_{\eta^0_{\pi,1}}} \mathbf{n}_{\eta^0_{\pi,1}} \ \text{ in } \Sigma_\infty^{s}, \\
& \hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0, L\} \times (0, \infty), \\
& \hat{\eta}_1(0) = \hat{\eta}_1^0 \ \text{ and } \ \hat{\eta}_2(0) = \hat{\eta}_2^0 \ \text{ in } \Gamma_s.
\end{aligned} \tag{1.2.7}
$$

where the operators C_π, $A_{1,1}$, A_2, A_3, $A_{4,1}$ are perturbations of the system that come from the linearization in a neighbourhood of the periodic solution and the changes of variables (see Chapter 4 for the precise definitions and notations).

We can now present our main result (Theorem 4.7.2). This theorem is written for the same linear system, in which the unknowns have been multiplied by $e^{\omega t}$ for some $-\omega < 0$.

Theorem 1.2.3. Let ω be positive. For all $(\mathbf{v}^0, \hat{\eta}_1^0, \hat{\eta}_2^0) \in \mathbf{H}_{cc}$ there exits

$$\mathbf{u}_c \in \mathcal{C}^0([0, +\infty); \mathbf{H}^2(\Gamma_i) \cap \mathbf{H}^1_0(\Gamma_i)) \cap \mathcal{C}^1([0, +\infty); \mathbf{L}^2(\Gamma_i)),$$

satisfying

$$\|\mathbf{u}_c(t)\|_{\mathbf{H}^2(\Gamma_i) \cap \mathbf{H}^1_0(\Gamma_i)} + \|\mathbf{u}_{c,t}(t)\|_{\mathbf{L}^2(\Gamma_i)} \leq K_1 e^{-\omega t}, \ \forall t > 0,$$

with $K_1 > 0$, such that the classical solution $(\mathbf{v}, q, \hat{\eta}_1, \hat{\eta}_2)$ to (1.2.7), which belongs to

- $\mathbf{v} \in \mathcal{C}^0([0, +\infty); \mathbf{L}^2(\Omega_{\pi,0})) \cap \mathcal{C}^0((0, +\infty); \mathbf{H}^2(\Omega_{\pi,0})) \cap \mathcal{C}^1((0, +\infty); \mathbf{L}^2(\Omega_{\pi,0}))$,
- $q \in \mathcal{C}^0((0, +\infty); H^1(\Omega_{\pi,0}))$,
- $\hat{\eta}_1 \in \mathcal{C}^0([0, +\infty); H^2_0(\Gamma_s)) \cap \mathcal{C}^0((0, +\infty); H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap \mathcal{C}^1((0, +\infty); H^2_0(\Gamma_s))$,
- $\hat{\eta}_2 \in \mathcal{C}^0([0, +\infty); L^2(\Gamma_s)) \cap \mathcal{C}^0((0, +\infty); H^2_0(\Gamma_s)) \cap \mathcal{C}^1((0, +\infty); L^2(\Gamma_s))$,

satisfies, for all $a > 0$ and some $K_2 \geq 0$ depending on a

$$\|(\mathbf{v}(t), \hat{\eta}_1(t), \hat{\eta}_2(t))\|_{\mathbf{H}^2(\Omega_{\pi,0}) \times H^4(\Gamma_s) \times H^2(\Gamma_s)} \leq K_2 e^{-\omega t}, \ \forall t > a.$$

In the previous statement the space $\mathbf{H}_{cc}$ is the space gathering the compatibility conditions on the initial data (see (4.7.2) for a precise definition).

Before presenting the general strategy developed to establish Theorem 1.2.3, let us add some remarks on the smallness hypothesis of the difference $\eta_\pi(t) - \eta_\pi(s)$. If the periodic solution is assumed to be small, the system can be linearized near the zero solution. Indeed, using a fixed point argument, it can be proved that the feedback obtained with this linearization stabilizes the fluid–structure system around any small solution.

On the contrary, without any smallness assumptions on the periodic solution, the spectrum of the linear system that we obtain after the change of variables cannot be computed. For instance, the linear fluid equations involve perturbations proportional to $\Delta\mathbf{u}$.

Our hypothesis on $\eta_\pi(t) - \eta_\pi(s)$ is an intermediary case between these two. It is more general than simply assuming that η_π is small, but it allows us to compute the spectrum of the underlying linear system.

The strategy to study (1.2.7) is as follows.

Construction of the parabolic evolution operator.

As done throughout this thesis, we begin with a reformulation of the linear system as a matrix evolution equation. The unbounded operator $\mathcal{A}(t)$ that we obtain has a domain that depends on time. This time dependency comes from the compatibility condition

$$\Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) \in V \cap \mathbf{H}^2(\Omega_{\pi,0}), \tag{1.2.8}$$

where both A_3 and L_1 depend on time (see (4.5.2)). This compatibility condition is fact a compatibility condition on the trace of the function $\Pi\mathbf{u}$ and the function $\Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2)$.

This time dependency of the domain noticeably complicates the construction of the parabolic evolution operator (this operator is the equivalent to the semigroup in the stationary theory). Fortunately, in [4], a theory is developed when the domain of the operator depends on time. This theory requires a suitable interpolated space to be independent of t. In that case, a parabolic evolution operator is constructed, with a regularity subspace corresponding to the interpolated space that is independent of t.

In our case, the trace condition can be removed when considering the interpolation, with a suitable parameter, between $\mathcal{D}(\mathcal{A}(t))$ and the ambient space $\mathbf{H}$. A proper balance is required between the Hölder regularity of the periodic solution and the parameter used to remove the trace condition in the interpolated spaces. We also faced challenging questions regarding the time-dependency of the norm of the interpolated space. These questions are solved in Theorem 4.6.2.

Despite the time-dependency of $\mathcal{D}(\mathcal{A}(t))$, we prove in Theorem 4.6.1 that the graph norm on $\mathcal{D}(\mathcal{A}(t))$ is equivalent to the norm of a fixed Banach space $\mathbf{D}$. This property is decisive for the stabilization of the system in strong norm.

Floquet theory: Hautus criteria for periodic system

After proving the existence of a parabolic evolution operator for our system, we can use the Floquet theory to study the stabilization. The general idea is to use the eigenvalues of the so-called Poincaré operator to split the state space into a stable part and an unstable one. We then adapt the stabilizability results proved in [42], using the uniform-in-time equivalence between the graph norm of $\mathcal{D}(\mathcal{A}(t))$ and the $\mathbf{D}$-norm.

Stabilization of the system.

Finally we study the stabilization of the system. The matrix formulation of the problem involves the derivative of the control. To solve this issue, and following [44], we include the control variables in the unknown of the problem by considering an extended system. Precisely, we search the control $\mathbf{u}_c$ under the form $\mathbf{u}_c^1 + \mathbf{u}_c^2$ with

$$\mathbf{u}_{c,t}^1 - 2\Delta_z \mathbf{u}_c^1 = \mathbf{g}_1,$$
$$\mathbf{u}_{c,t}^2 - \Delta_z \mathbf{u}_c^2 = \mathbf{g}_2,$$

where Δ_z is the one dimensional Laplace operator on the flat boundary Γ_i. The variables $\mathbf{u}_c^1$ and $\mathbf{u}_c^2$ are included in the extended system and the new control variable is $(\mathbf{g}_1, \mathbf{g}_2)$. Two ideas are applied here:

- The heat operator is used to ensure that the extended operator (4.7.6) still possesses a parabolic evolution operator.
- The control variable is ‘doubled’, and two heat equations with two different diffusive coefficients are introduced to increase the ‘richness’ of the control. This is instrumental in proving the unique continuation property for the extended system.

Using these ideas and the unique continuation result in [21, 22] we prove that the time-dependent Hautus test is satisfied, which ensures the stabilization of the linear system.

1.3 Toolbox

In this section we gather some tools and key techniques used throughout the thesis.

1.3.1 Configuration of the fluid domain

In fluid mechanics two techniques are commonly used to describe a fluid flow. Eulerian coordinates associate to each point of the fluid a velocity vector. On the contrary, the

Lagrangian coordinates are associated with a fluid particle and follow it through its evolution.

One of the difficulties in the analysis of fluid–structure systems comes from the moving domain. In the system that we consider, the structure equation is written in a reference configuration, which corresponds to a Lagrangian description, whereas the fluid is described in Eulerian coordinates.

The usual technique to fix the fluid domain is to parametrise the fluid equations by the flow. Consider a fluid flow written in Eulerian coordinates in a moving domain $\Omega(t)$. The flow associated to the fluid velocity $\mathbf{u}$ is a map $X(\cdot, t)$ from $\Omega(0)$ to $\Omega(t)$ satisfying the differential equation

$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t) = \mathbf{u}(X(y,t),t) \text{ for all } y \in \Omega(0), \\ X(y,0) = y \text{ for all } y \in \Omega(0). \end{cases}$$

We then consider the parametrisation $\overline{\mathbf{u}}(y,t) = \mathbf{u}(X(y,t),t)$. The function $\overline{\mathbf{u}}$ is now defined in the fixed domain $\Omega(0)$. This change of variables corresponds to the change of viewpoint from the Eulerian to the Lagrangian description of the fluid.

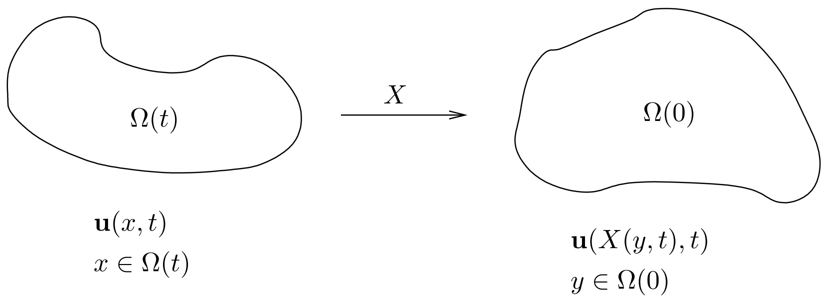

Figure 1.2: Eulerian/Lagrangian configuration

The fluid–structure system that we consider in this thesis is initially written in the domain $\Omega_{\eta(t)}$ described in Figure 1.1. Due to the specificity of the domain, of 'rectangular type', and the regularity of the function η, the domain $\Omega_{\eta(t)}$ can be fixed with a smoother change of variables that the one provided by the flow.

The classical change of variables performed for this system is

$$\mathcal{T}_0(t) : \begin{cases} \Omega_{\eta(t)} \longrightarrow \Omega \\ (x,y) \mapsto \left(x, \frac{y}{1+\eta(x,t)}\right), \end{cases}$$

where $\Omega = (0,L) \times (0,1)$. This change of variables is the identity at time $T = 0$ if and only if the initial displacement of beam, denoted by $\eta(0) = \eta_1^0$, is equal to zero. This leads to some difficulties in the local-in-time existence of strong solutions: to apply the fixed point theorem on the complete nonlinear system, one has to assume that η_1^0 is small in an appropriate norm. The solution that we propose is to consider another change of variables, namely:

$$\mathcal{T}_{\eta_1^0}(t) : \left\{ \begin{array}{l} \Omega_{\eta(t)} \longrightarrow \Omega_{\eta(0)} \\ (x,y) \mapsto \left(x, \frac{1+\eta_1^0(x)}{1+\eta(x,t)} y\right), \end{array} \right.$$

This change of variables maps the moving domain $\Omega_{\eta(t)}$ onto the domain of the fluid $\Omega_{\eta(0)}$ at the time $T = 0$. At $T = 0$, this change of variable is the identity, this is used in Chapter 2 to ensure that the nonlinearities of the system are small for small T, which enables the proof of the existence result.

If η_1^0 is small the two changes of variables are similar. In this case, it is more practical to work in the rectangular domain Ω. This is the option chosen to prove the existence of strong solutions over an arbitrary time interval $[0, T]$ for a small enough initial beam displacement.

In Chapter 3 we study the existence of periodic solutions for the fluid–structure system with T-periodic source terms on the inflow and the outflow. Smallness assumptions on the source terms are used to construct a periodic solution and we can choose the rectangular domain Ω as a reference configuration.

A non-rectangular reference configuration is used again in Chapter 4 for the stabilization of the fluid–structure system in a neighbourhood of a periodic solution. The perturbed solution is written in a domain $\Omega_{\eta(t)}$ and the periodic solution on a domain $\Omega_{\eta_\pi(t)}$. To compare the two solutions we perform a first change of variables mapping $\Omega_{\eta(t)}$ onto $\Omega_{\eta_\pi(t)}$:

$$\mathcal{T}_{\eta_\pi}(t) : \left\{ \begin{array}{l} \Omega_{\eta(t)} \longrightarrow \Omega_{\eta_\pi(t)} \\ (x,y) \mapsto \left(x, \frac{1+\eta_\pi(x,t)}{1+\eta(x,t)} y\right). \end{array} \right.$$

The perturbed solution is then defined in the domain $\Omega_{\eta_\pi(t)}$ and can be compared with the periodic solution. However, the domain still depends on the time. The function η_π, describing the geometry of the domain, is not an unknown of the system since it represents the (beam displacement of the) given periodic solution around which we want to stabilise the perturbed solution. Fixing the domain $\Omega_{\eta_\pi(t)}$ by another change of variables therefore does not introduce any additional nonlinear terms but the underlying linear system is strongly perturbed. In our analysis we assume that the periodic beam displacement η_π stays in a small 'cylinder', i.e., $\eta_\pi(t) - \eta_\pi(s)$ is small for all t and s. We use a change of variables mapping $\Omega_{\eta_\pi(t)}$ into $\Omega_{\eta_{\pi,1}^0}$, where $\eta_{\pi,1}^0 = \eta_\pi(0)$:

$$\mathcal{T}_{\eta_{\pi,1}^0}(t) : \left\{ \begin{array}{l} \Omega_{\eta(t)} \longrightarrow \Omega_{\eta_\pi(0)} \\ (x,y) \mapsto \left(x, \frac{1+\eta_{\pi,1}^0(x)}{1+\eta(x,t)} y\right). \end{array} \right.$$

With this change of variables, the additional problematic linear terms involve the small difference $\eta_\pi(t) - \eta_\pi(0)$, which enable their treatment in a similar way as the nonlinear terms. The different changes of variables used in Chapter 4 are summarised in Figure 1.3.

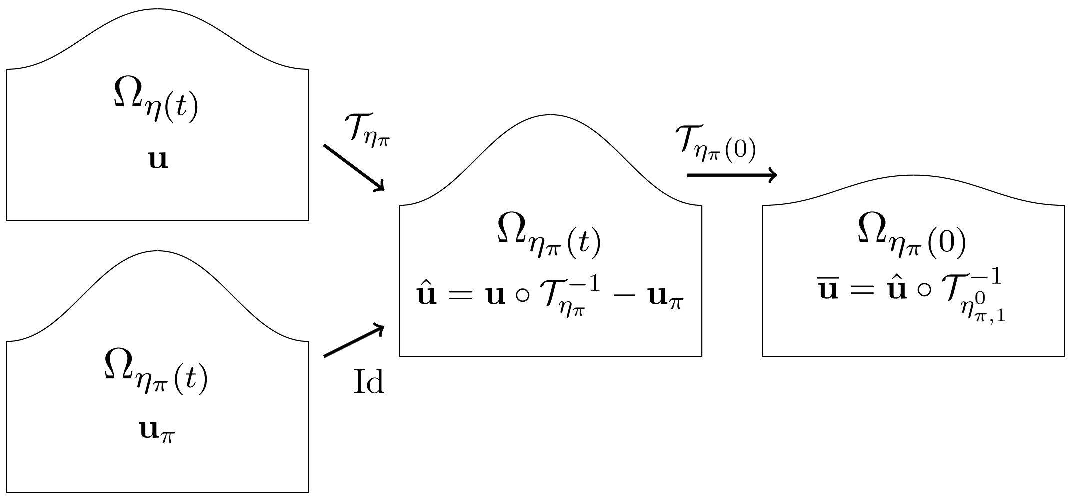

Figure 1.3: The changes of variables used in Chapter 4. $(\mathbf{u}_\pi, \eta_\pi)$ are the velocity and beam displacement of the periodic solution around which we want to stabilise the perturbed solution, with velocity and beam displacement $(\mathbf{u}, \eta)$.

1.3.2 Fluid equations

Stokes system

Consider the Stokes system in the rectangular domain Ω,

$$
\begin{aligned}
&-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \ \text{ in } \Omega, \\
&\mathbf{u} = 0 \text{ on } \Gamma_s, \ \ \mathbf{u} = 0 \text{ on } \Gamma_b,
\end{aligned} \tag{1.3.1}
$$

with either one of the following boundary conditions on the inflow and the outflow:

1. $\mathbf{u} = 0$ on $\Gamma_i \cup \Gamma_o$.
2. $u_2 = 0$ and $p = 0$ on $\Gamma_i \cup \Gamma_o$.
3. $\sigma(\mathbf{u}, p)\mathbf{n} = 0$ on $\Gamma_i \cup \Gamma_o$.

If $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the expected regularity for a pair $(\mathbf{u}, p)$ solution to (1.3.1) is $\mathbf{H}^2(\Omega) \times H^1(\Omega)$. This expected regularity is however challenged by geometry of the domain, that has corners, and by the mixed boundary conditions in case (2) or (3).

Case (1): For homogeneous Dirichlet boundary conditions on the velocity at the inflow and the outflow, the solution $(\mathbf{u}, p)$ of (1.3.1) belongs to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$. This results is well known and can be found, for example, in the book of Maz'ya and Rossmann [47].

For non-homogeneous boundary conditions, continuity compatibility conditions are required at the corners of the domain. In Chapter 3 and 4, the Dirichlet boundary condition on the inflow is taken in $\mathbf{H}_0^{3/2}(\Gamma_i) = \mathbf{H}^{3/2}(\Gamma_i) \cap \mathbf{H}_0^1(\Gamma_i)$ to fulfil these compatibility conditions.

Case (2): One of the main contributions of this thesis is the study of (1.3.1) with a junction between Dirichlet boundary conditions on the velocity and pressure boundary conditions. We shall see in Chapter 2 that the pair $(\mathbf{u}, p)$ once again belongs to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$.

Inside the domain and in a neighbourhood of flat parts of the boundary, the $\mathbf{H}^2$-regularity can be proved using cut-off functions and known results for Stokes with Dirichlet boundary conditions on the velocity. The interesting issue is the behaviour of the solution around the corners. The main idea to tackle this issue is to use a symmetry argument to extend (1.3.1) to a larger domain. The corners then 'disappear' in the extended domain and we can use local regularity results to recover the $\mathbf{H}^2$-regularity for the fluid. As explained in the previous section, the geometry of the domain that we consider is usually more complicated than the rectangular Ω and this symmetry technique therefore needs to be adapted to the curved geometry of the domain.

This adaptation requires in particular to find a solution to the following lifting problem:

$$\begin{aligned} &\operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \\ &\mathbf{w} = g\,\mathbf{e}_2 \text{ on } \Gamma_s, \\ &\mathbf{w} = 0 \text{ on } \Gamma_b, \\ &w_2 = 0 \text{ on } \Gamma_i \cup \Gamma_o. \end{aligned} \tag{1.3.2}$$

We prove that this equation can be solved for functions g belonging to $H_{00}^{3/2}(\Gamma_s)$. This strict subspace of $H_0^{3/2}(\Gamma_s)$ was introduced in the works of Lions and Magenes [38]. It corresponds to the 1/2-interpolate space between $H_0^1(\Gamma_s)$ and $H_0^2(\Gamma_s)$. Our interest in this space comes from its properties with regards to the symmetry that is applied to remove the corners. A function in $H_{00}^{3/2}(\Gamma_s)$ preserves its $H^{3/2}$-regularity after an odd or an even symmetry. This property is not satisfied by functions in $H_0^{3/2}(\Gamma_s)$ for even symmetry.

Case (3): Unlike the previous case, for mixed boundary conditions with a junction Neumann/Dirichlet the solution of (1.3.1) does not belong to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$. It is shown in [47] that $(\mathbf{u}, p)$ belongs to weighed Sobolev spaces $\mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega)$ where δ is a non negative parameter describing the 'explosion' of the $\mathbf{H}^2$-norm around the corners.

Expression of the pressure

The treatment of the pressure is a key point for incompressible fluid equations. The pressure of the fluid can be seen as the Lagrange multiplier related to the constrain div $\mathbf{u} = 0$. Two different approaches are possible. A first one consists in keeping the pressure in the variational formulation of (1.3.1), which leads to the so-called saddled point formulation of the problem. The second approach consists in removing the pressure from the fluid equations using a projector to 'kill' the gradient of H^1-functions. This projector is called the Leray projector and corresponds to the Helmholtz decomposition in L^2-spaces,

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_n(\Omega) \oplus \nabla H^1(\Omega),$$

where $\mathbf{V}^0_n(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \text{div } \mathbf{v} = 0 \text{ in } \Omega, \, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. The previous decomposition is used to study (1.3.1) with Dirichlet boundary conditions (1). For mixed boundary conditions (2) and (3), the appropriate decomposition is

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_{n,\Gamma_d}(\Omega) \oplus \nabla H_{\Gamma_{i,o}}(\Omega),$$

where $\Gamma_{i,o} = \Gamma_i \cup \Gamma_o$, $\Gamma_d = \Gamma_s \cup \Gamma_b$ and

- $\mathbf{V}^0_{n,\Gamma_d}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \text{div } \mathbf{v} = 0 \text{ in } \Omega, \, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_d\}$.
- $H^1_{\Gamma_{i,o}}(\Omega) = \{q \in H^1(\Omega) \mid q = 0 \text{ on } \Gamma_{i,o}\}$.

The Leray projector $\Pi : \mathbf{L}^2(\Omega) \to \mathbf{V}^0_{n,\Gamma_d}(\Omega)$ associated with this decomposition is described through the solutions to two elliptic equations. For $\mathbf{u} \in \mathbf{L}^2(\Omega)$, $\Pi\mathbf{u} = \mathbf{u} - \nabla p_{\mathbf{u}} - \nabla q_{\mathbf{u}}$ with

$$\begin{aligned} &p_{\mathbf{u}} \in H^1_0(\Omega), \;\; \Delta p_{\mathbf{u}} = \text{div } \mathbf{u} \in H^{-1}(\Omega), \\ &q_{\mathbf{u}} \in H^1_{\Gamma_{i,o}}(\Omega), \;\; \Delta q_{\mathbf{u}} = 0, \;\; \frac{\partial q_{\mathbf{u}}}{\partial \mathbf{n}} = (\mathbf{u} - \nabla p_{\mathbf{u}}) \cdot \mathbf{n} \text{ on } \Gamma_d, \;\; q_{\mathbf{u}} = 0 \text{ on } \Gamma_{i,o}. \end{aligned} \tag{1.3.3}$$

One could consider combining these two equations in one, writing $\Pi\mathbf{u} = \mathbf{u} - \nabla\gamma_{\mathbf{u}}$ with $\gamma_{\mathbf{u}} = p_{\mathbf{u}} + q_{\mathbf{u}}$ solution to

$$\gamma_{\mathbf{u}} \in H^1_{\Gamma_{i,o}}(\Omega), \;\; \Delta\gamma_{\mathbf{u}} = \text{div } \mathbf{u}, \;\; \frac{\partial \gamma_{\mathbf{u}}}{\partial \mathbf{n}} = \mathbf{u} \cdot \mathbf{n} \text{ on } \Gamma_d, \;\; \gamma_{\mathbf{u}} = 0 \text{ on } \Gamma_{i,o}. \tag{1.3.4}$$

However, this equation (1.3.4) is only well-posed if the normal trace $\mathbf{u} \cdot \mathbf{n}$ of $\mathbf{u}$ is properly defined, which requires $\mathbf{u}$ to be in a more regular space than $\mathbf{L}^2(\Omega)$ (typically, div $\mathbf{u}$ must belong to $L^2(\Omega)$). For $\mathbf{u} \in \mathbf{L}^2(\Omega)$, however, all the terms in (1.3.3) make sense since div $(\mathbf{u} - \nabla p_{\mathbf{u}}) = 0$, which ensures that the normal trace $(\mathbf{u} - \nabla p_{\mathbf{u}}) \cdot \mathbf{n}$ is defined in $(H^{1/2}(\partial\Omega))'$. We note from (1.3.4) that, if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ and div $\mathbf{u} \in L^2(\Omega)$, $(I - \Pi)\mathbf{u}$ depends only on the divergence of $\mathbf{u}$ and its normal trace on Γ_d.

In the fluid–structure model that we consider the pressure of the fluid appears in the structure equation written on Γ_s. To remove this pressure, it must be expressed in terms of other unknowns $\mathbf{u}$ and η, which can be achieved using the Leray projector. To simplify the exposition, consider the unsteady Stokes system with boundary condition (2). Applying $(I - \Pi)$ to the first line $\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = 0$ leads to

$$\nabla p = \nu(I - \Pi)\Delta\mathbf{u} - (I - \Pi)\mathbf{u}_t$$

(note that $(I - \Pi)\nabla p = \nabla p$ since $p \in H^1_{\Gamma_{i,o}}(\Omega)$). Using (1.3.4), we have $\nu(I - \Pi)\Delta\mathbf{u} = \nu(I - \Pi)\Delta\Pi\mathbf{u} = \nabla\rho$ with

$$\begin{aligned} &\Delta\rho = 0 \text{ in } \Omega,\\ &\frac{\partial\rho}{\partial\mathbf{n}} = \nu\Delta\Pi\mathbf{u}\cdot\mathbf{n} \text{ on } \Gamma_d,\\ &\rho = 0 \text{ on } \Gamma_{i,o}. \end{aligned}$$

Suppose that $\mathbf{u} = \eta_t\mathbf{e}_2$ on Γ_s. Then, since $\mathbf{e}_2 = \mathbf{n}$ on Γ_s, $(I - \Pi)\mathbf{u}_t = \nabla q_t$ with

$$\begin{aligned} &\Delta q = 0 \text{ in } \Omega,\\ &\frac{\partial q}{\partial\mathbf{n}} = \eta_t \text{ on } \Gamma_s,\\ &\frac{\partial q}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_b,\\ &q = 0 \text{ on } \Gamma_{i,o}. \end{aligned}$$

The pressure is then given by $p = \rho - q_t$. This splitting method requires $\mathbf{u}$ to be in $\mathbf{H}^2(\Omega)$ to define the normal trace of $\Delta\Pi\mathbf{u}$ on Γ_d (this trace is indeed well defined since $\Delta\Pi\mathbf{u}$ belongs $\mathbf{L}^2(\Omega)$ and has divergence $\operatorname{div}\,\Delta\Pi\mathbf{u} = \Delta\operatorname{div}\,\Pi\mathbf{u} = 0$). This decomposition is used in Chapter 2 and 3.

Another approach, developed in a recent paper of Fournié, Ndiaye and Raymond [23], consists in multiplying the fluid equations with a suitable test function, and to use a duality technique to recover an expression of the pressure. The main advantage of this method is its applicability even if $\mathbf{u}$ does not belong to $\mathbf{H}^2(\Omega)$. It was used by the authors to deal with a fluid–structure system involving mixed boundary conditions (3). This method is also more 'compact' than the traditional one, and will thus be preferred in Chapter 4 in which the fluid equations (after two changes of variables and with time-dependent coefficients) presents many additional terms.

1.3.3 Structure equation

Throughout this thesis the structure is described by a damped Euler–Bernoulli equation whose eigenvalue problem takes the following form (in which the time derivatives are replaced with multiplications by $\lambda \in \mathbb{C}$):

$$\begin{aligned}&\lambda\eta_1 - \eta_2 = F_1 \text{ on } \Gamma_s,\\ &\lambda\eta_2 - \beta\eta_{1,xx} - \gamma\eta_{2,xx} + \alpha\eta_{1,xxxx} = F_2 \text{ on } \Gamma_s,\\ &\eta_1 = 0 \text{ and } \eta_{1,x} = 0 \text{ on } \{0, L\}.\end{aligned} \tag{1.3.5}$$

Here, β, γ, α are non negative parameters relative to the structure.

The regularity of the solutions to (1.3.5) depends on the (strict) positivity of the damped coefficient γ and of the elastic coefficient α. In our study, we work in the most regular framework and we suppose that $\gamma > 0$ and $\beta > 0$. A complete discussion about these coefficients, and consequences of removing the positivity assumption, can be found in the recent work of Grandmont, Hillairet and Lequeurre [29].

The matrix form of (1.3.5) is

$$\lambda \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}0 & I\\ A_{\alpha,\beta} & \gamma\Delta\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix}F_1\\ F_2\end{pmatrix}, \tag{1.3.6}$$

where $A_{\alpha,\beta} = \beta\Delta - \alpha\Delta^2$ with $\Delta = \partial_{xx}$. The associated unbounded operator $\mathcal{A}_b$ in $H_b = H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ defined by

$$\mathcal{D}(\mathcal{A}_b) = \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s), \;\; \mathcal{A}_b = \begin{pmatrix}0 & I\\ A_{\alpha,\beta} & \gamma\Delta\end{pmatrix},$$

is studied, in a more general framework, in the paper of Chen and Triggiani [16]. In particular we have the following result.

Proposition 1.3.1. The operator $(\mathcal{A}_b, \mathcal{D}(\mathcal{A}_b))$ is the infinitesimal generator of an analytic semigroup.

This proposition enables the usage of the so-called isomorphism theorem, in the Hilbertian theory of analytic semigroup, to recover maximal regularity results on the evolution equation

$$\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}0 & I\\ A_{\alpha,\beta} & \gamma\Delta\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix}F_1\\ F_2\end{pmatrix}, \quad \begin{pmatrix}\eta_1(0)\\ \eta_2(0)\end{pmatrix} = \begin{pmatrix}\eta_1^0\\ \eta_1^0\end{pmatrix}. \tag{1.3.7}$$

If (F_1, F_2) belongs to H_b and (η_1^0, η_2^0) belongs to $\big(H^3(\Gamma_s) \cap H_0^2(\Gamma_s)\big) \times H_0^1(\Gamma_s)$ then (1.3.7) admits a unique solution $(\eta_1, \eta_2) \in L^2(0, T; \mathcal{D}(\mathcal{A}_b)) \cap H^1(0, T; H_b)$.

1.3.4 Analytic semigroups

The theory of analytic semigroups is intensively used in this thesis. The semigroup reformulation of the fluid–structure system is initially used to decouple the system. Additionally, it gives access to the powerful tools of this theory. This framework is used:

- In Chapter 2 to obtain maximal regularity results on the linear system.
- In Chapter 3 to construct a time-periodic solution of the linear system.
- In Chapter 4 to prove the existence of classical solutions and to use the Floquet theory to stabilize the system.

We introduce some notations, mostly used in Chapter 4. Let (E_0, E_1) be a pair of densely embedded Banach spaces $E_1 \overset{d}{\hookrightarrow} E_0$. Using the notations in [4], let

$$\mathcal{H}(E_1, E_0),$$

be the set of all linear operators $A \in \mathcal{L}(E_1, E_0)$ such that A, considered as an unbounded operator in E_0 with domain E_1, is the infinitesimal generator of a strongly continuous analytic semigroup $(S(t))_{t\geq 0}$ on $\mathcal{L}(E_0)$. Different definitions of analytic semigroups exists. In our case we say that a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $\mathcal{L}(E_0)$ with infinitesimal generator A is analytic if there exits a sector $S_{a,\theta} = \{z \in \mathbb{C} \mid \lambda \neq a,\ |\arg(\lambda - a)| < \pi/2 + \theta\}$ with $a \in \mathbb{R}$ and $\theta \in (0, \frac{\pi}{2})$, such that $\{a\} \cup S_{a,\theta} \subset \rho(A)$ and

$$\exists M > 0, \ \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(E_0)} \leq \frac{M}{|\lambda - a|}, \ \ \forall \lambda \in S_{a,\theta}.$$

An operator A satisfying this estimate is said to be sectorial. The previous definition can be found in [10]. The theory of analytic semigroups can be developed without assuming that the semigroups are strongly continuous, as done for example in [43]. The difference between the two definitions lies in the density of the domain $\mathcal{D}(A)$. A sectorial operator A is the generator of a strongly continuous semigroup if and only if $\overline{\mathcal{D}(A)} = E_0$. For all these reasons some authors state that they are working with strongly continuous analytic semigroups, see for example [4]. As we will always assume that $\overline{\mathcal{D}(A)} = E_0$, the strong continuity property is directly included in our definition of analytic semigroups, as in [10, 52]. An equivalent definition of $(A, \mathcal{D}(A))$ being the infinitesimal generator of an analytic semigroup, used in [34]: A is densely defined, closed and there exists $\lambda_0 \in \mathbb{R}$ such that

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \lambda_0\} \subset \rho(A), \tag{1.3.8}$$

and $M > 0$ with:

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(E_0)} \leq \frac{M}{1 + |\lambda|} \text{ for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq \lambda_0. \tag{1.3.9}$$

For $M > 0$ and $\lambda_0 \in \mathbb{R}$, we denote by $\mathfrak{A}_{M,\lambda_0}(E_0)$ the set of all densely defined and closed linear operators A in E_0 such that (1.3.8) and (1.3.9) hold.

We now present two fundamental results provided by the theory of analytic semigroups. Let $A \in \mathcal{H}(E_1, E_0)$. For $T > 0$, consider the Cauchy problem

$$\begin{aligned} &u'(t) = Au(t) + f(t),\ t \in (0, T), \\ &u(0) = u^0. \end{aligned} \tag{1.3.10}$$

When E_0 and E_1 are Hilbert spaces, the following so-called isomorphism theorem ensures the existence of strict solutions in L^2 for this problem.

Theorem 1.3.1. The map

$$\begin{cases} L^2(0,T;E_1) \cap H^1(0,T;E_0) \longrightarrow L^2(0,T;E_0) \times [\mathcal{D}(A), E_0]_{1/2} \\ u \mapsto (u' - Au, u(0)), \end{cases}$$

is an isomorphism. In particular, for all $f \in L^2(0,T;E_0)$ and $u^0 \in [\mathcal{D}(A), E_0]_{1/2}$, the Cauchy problem (1.3.10) has a unique strict solution in $L^2(0,T;E_0)$.

This theorem can be found in [10, Part II, Section 1.3, Theorem 3.1]. Remark that we can take $T = +\infty$ if and only if the semigroup generated by A is of negative type. In that case, the system (1.3.10) is exponentially stable.

Concerning the existence of solutions in a continuous framework the following theorem can be found in [4].

Theorem 1.3.2. Suppose that $u^0 \in E_0$ and $f \in \mathcal{C}^\rho([0,T];E_0)$ for some $\rho \in (0,1)$. Then the Cauchy problem (1.3.10) has a unique classical solution $u \in \mathcal{C}^0([0,T];E_0) \cap \mathcal{C}^\rho((0,T];E_1) \cap \mathcal{C}^{1+\rho}((0,T];E_0)$. If $u^0 \in E_1$ then this solution is strict, i.e., u belongs to $\mathcal{C}^0([0,T];E_1) \cap \mathcal{C}^1([0,T];E_0)$.

This theorem is used in Chapter 3 to recover the Hölder regularity of the periodic solution. Remark that, in the previous theorem, the solution u is not Hölder continuous at $t = 0$. A sufficient condition to obtain this continuity is to assume that $Au^0 + f(0) \in [E_1, E_0]_\rho$, see [43, Theorem 4.3.1].

Chapter 2

Fluid–structure system with boundary conditions involving the pressure

2.1 Introduction

2.1.1 Setting of the problem

We study the coupling between the 2D Navier–Stokes equations and a damped Euler–Bernoulli beam equation in a rectangular type domain, where the beam is a part of the boundary. Let $T > 0$, $L > 0$ and consider the spatial domain Ω in $\mathbb{R}^2$ defined by $\Omega = (0, L) \times (0, 1)$. Let us set $\Gamma_i = \{0\} \times (0, 1)$ and $\Gamma_o = \{L\} \times (0, 1)$ the left and right boundaries, $\Gamma_s = (0, L) \times \{1\}$, $\Gamma_b = (0, L) \times \{0\}$ and $\Gamma = \partial\Omega$ the boundary of Ω. Let η be the displacement of the beam. The function η is defined on $\Gamma_s \times (0, T)$ with values in $(-1, +\infty)$. Let $\Omega_{\eta(t)}$ and $\Gamma_{\eta(t)}$ be the sets defined by

$$\Omega_{\eta(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ 0 < y < 1 + \eta(x, 1, t)\},$$
$$\Gamma_{\eta(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ y = 1 + \eta(x, 1, t)\}.$$

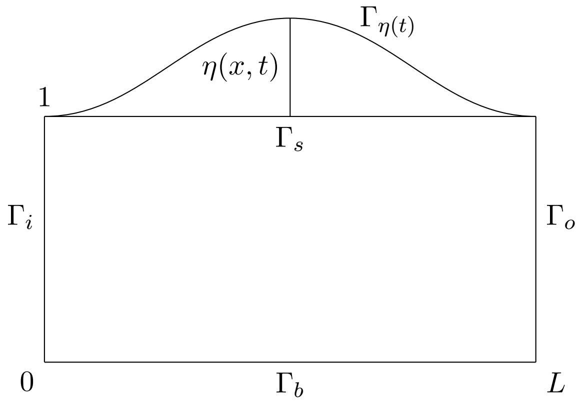

Figure 2.1: fluid–structure system.

We also set $\Gamma_{i,o} = \Gamma_i \cup \Gamma_o$. The space-time domains are denoted by

$$\Sigma_T^s = \Gamma_s \times (0,T),\ \Sigma_T^{i,o} = \Gamma_{i,o} \times (0,T),\ \Sigma_T^b = \Gamma_b \times (0,T),$$
$$\widetilde{\Sigma}_T^s = \bigcup_{t\in(0,T)} \Gamma_{\eta(t)} \times \{t\},\ \ \widetilde{\mathcal{Q}}_T = \bigcup_{t\in(0,T)} \Omega_{\eta(t)} \times \{t\}.$$

We study the following fluid structure system coupling the Navier–Stokes equations and the damped Euler–Bernoulli beam equation

$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\,\sigma(\mathbf{u},p) = 0,\ \ \operatorname{div}\,\mathbf{u} = 0\ \text{ in } \widetilde{\mathcal{Q}}_T,\\
&\mathbf{u}(x,y,t) = \eta_t(x,1,t)\mathbf{e}_2\ \text{ for } (x,y,t) \in \widetilde{\Sigma}_T^s,\\
&u_2 = 0\ \text{ and }\ p + (1/2)|\mathbf{u}|^2 = 0\ \text{ on } \Sigma_T^{i,o},\\
&\mathbf{u} = 0\ \text{ on } \Sigma_T^b,\ \ \mathbf{u}(0) = \mathbf{u}^0\ \text{ in } \Omega_{\eta_1^0},\\
&[\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx}](x,1,t)\\
&\qquad = \psi[\mathbf{u},p,\eta](x,1+\eta(x,1,t),t)\ \text{ for } (x,t) \in (0,L)\times(0,T),\\
&\eta(\cdot,1,\cdot) = 0\ \text{ and }\ \eta_x(\cdot,1,\cdot) = 0\ \text{ on } \{0,L\}\times(0,T),\\
&\eta(\cdot,0) = \eta_1^0\ \text{ and }\ \eta_t(\cdot,0) = \eta_2^0\ \text{ in } \Gamma_s,
\end{aligned} \tag{2.1.1}$$

where $\mathbf{u}$ is the velocity, p the pressure, η the displacement of the beam and

$$\begin{aligned}
\sigma(\mathbf{u},p) &= -pI + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T),\\
\psi[\mathbf{u},p,\eta](x,y,t) &= -\sigma(\mathbf{u},p)(x,y,t)(-\eta_x(x,1,t)\mathbf{e}_1 + \mathbf{e}_2)\cdot\mathbf{e}_2,
\end{aligned}$$

for all $(x,y,t) \in \widetilde{\Sigma}_T^s$. For a function f defined on the flat domain Γ_s or on $(0,L)$ we use the following abuse of notation : $f(x) = f(x,1) = f(x,y)$ for $(x,y) \in (0,L)\times\mathbb{R}$. This notation will typically be used for $f = \eta$ or $f = \psi[\mathbf{u}, p, \eta]$. Hence the beam equation can be written

$$\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = \psi[\mathbf{u}, p, \eta] \text{ on } \widetilde{\Sigma}^s_T.$$

In the previous statement $\mathbf{e}_1 = (1,0)^T$, $\mathbf{e}_2 = (0,1)^T$, $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, $\nu > 0$ is the constant viscosity of the fluid and ψ is a force term modelling the interaction between the fluid and the beam (see [56], [49]). The constant $\beta \geq 0$, $\gamma > 0$ and $\alpha > 0$ are parameters relative to the structure. This system can be used to model the blood flow through human arteries, provided that the arteries are large enough (see [49]). The homogeneous Dirichlet boundary condition on Γ_b is used to simplify the presentation; the same system with two beams can be studied in the same way.

The existence of weak solutions to system (2.1.1) is proved in [49]. Here we would like to prove the existence of strong solutions for the same system. A similar system is studied in [37] with Dirichlet inflow and outflow boundary conditions, and in [28] with periodic inflow and outflow boundary conditions. In [37] a local in time existence of strong solutions is proved without smallness assumptions on $\mathbf{u}^0$ and η_2^0. The initial condition η_1^0 is not zero, but, as far as we understand, the proof in [37] is valid only if η_1^0 is small enough (see below). Since some results in [28] rely on the techniques of [37], it seems that the global existence result of [28] is also only valid if η_1^0 is small. The existence of strong solutions to the fluid–structure system with a non-small η_1^0 therefore still seems to be an open question. The present paper brings an answer to this question, by establishing the local-in-time existence of strong solutions without smallness conditions on $\mathbf{u}^0$, η_1^0 and η_2^0.

We prove this result for (2.1.1), that is to say with boundary conditions involving the pressure. However the issue raised by a non-small η_1^0 is purely a nonlinear one, whose treatment is independent of the boundary conditions (once the proper regularity results for the linearized system have been established). The technique developped here, based on a novel change of variables, therefore fills the gap in [37]. The existence of strong solutions for (2.1.1) relies on regularity results of the underlying Stokes system and Leray projector. Three elements challenge this regularity here: the change of variable used to deal with a generic η_1^0, the corners of the domain, and the junctions between Dirichlet and pressure boundary conditions. To overcome these challenges, we use symmetry techniques and a minimal-regularity transport of H^3 functions. We note that, for smooth domains (no corner, no minimal-regularity change of variables), the regularity result for the Stokes system was established in [11, 12]. As a by-product of our analysis, we also obtain the existence over an arbitrary time interval $[0, T]$ of strong solutions to system (2.1.1), provided that the initial data $\mathbf{u}^0$, η_1^0 and η_2^0 are small enough.

Let us detail the gap mentioned above. In [37], a key estimate, obtained through interpolation techniques, is

$$\|\eta\|_{L^\infty(\Sigma^s_T)} + \|\eta_x\|_{L^\infty(\Sigma^s_T)} + \|\eta_{xx}\|_{L^\infty(\Sigma^s_T)} \leq CT^\chi \|\eta\|_{H^{4,2}(\Sigma^s_T)}, \tag{2.1.2}$$

for some $\chi > 0$ and $C > 0$ (see Section 2.3.1 for the functional spaces). If T goes to 0 the previous estimate implies that $\|\eta_0^1\|_{L^\infty(\Gamma_s)} = 0$ and thus $\eta_0^1 = 0$. A careful study of the interpolation techniques and the Sobolev embeddings used to prove (2.1.2) shows that the time dependency of the constants was omitted. The fundamental reason behind this issue is related to the change of variables, used to fix the domain to Ω, that introduces additional 'geometrical' nonlinearities. These nonlinearities are not small for small T if the change of variables is not the identity at $T = 0$. To solve this issue we rewrite the system (2.1.1) in the fixed domain $\Omega_{\eta_1^0}$ instead of Ω. The geometrical nonlinear terms now involve the difference $\eta - \eta_1^0$ which is small when T is small.

Since our technique fills the gap in [37], this means that the global-in-time existence result of [28] for periodic boundary conditions is now genuinely established without smallness assumption on η_1^0. An interesting question is to consider if the result in [28] can be adapted, starting from our local-in-time existence result, to obtain a global-in-time existence of solutions with non-standard boundary conditions involving the pressure. To do so, additional estimates should be proved to ensure that a collision between the beam and the bottom of the fluid cavity does not occur in finite time.

Finally we would like to mention some references related to our work. The Stokes and Navier–Stokes system with pressure boundary conditions was initially study in [17], using weak variational solutions. A first rigorous existence result for (2.1.1) with periodic boundary conditions goes back to [9] where an iterative method was used to handle the coupled system. The feedback stabilization of (2.1.1) with Dirichlet inflow and outflow boundary conditions is studied in [56] and provides a semigroup approach for the linearized system, based on a splitting of the pressure, that is used in the present article. This semigroup framework was already used in [5, 6] for a linear model.

2.1.2 Main results

The main result of this paper is Theorem 2.4.3 which proves the existence of a unique local strong solution for the fluid–structure system (2.1.1) without smallness assumptions on the initial data. We also state in Theorem 2.4.4 the existence of a unique strong solution on the time interval $[0, T]$ with $T > 0$ an arbitrary fixed time, for small enough initial data. Several changes of variables are done on (2.1.1) and these results are given for equivalent system (see (2.2.6)).

The structure of the article is as follows. In Section 2.2, we rewrite (2.1.1) in a fixed domain and we explain ideas of the proof which consist in studying a linear system associated with (2.1.1) and in using a fixed point argument. In Section 2.3 we eliminate the pressure in the beam equation by expressing it in terms the velocity. We then rewrite the system as an abstract evolution equation and we prove that the underlying operator generates an analytic semigroup. Finally we prove the nonlinear estimates, with explicit time dependency, in Section 2.4 and we conclude with a fixed point procedure. All this process is based on the extension to non-standard boundary conditions of known result on the Stokes equations. This is detailed in the appendix.

2.2 Plan of the paper

2.2.1 Equivalent system in a reference configuration

In order to study system (2.1.1), we are going to rewrite the system in a reference configuration which can be chosen arbitrarily. For that, throughout what follows, we choose a function η^0 belonging to $H^3(\Gamma_s) \cap H_0^2(\Gamma_s)$, and satisfying $1 + \eta^0(x) > 0$ for all $x \in (0, L)$. Set

$$\begin{aligned} &\Omega_0 = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,L),\ 0 < y < 1 + \eta^0(x)\}, \qquad && Q_T = \Omega_0 \times (0,T), \\ &\Gamma_0 = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,L),\ y = 1 + \eta^0(x)\}, && \Sigma_T^0 = \Gamma_0 \times (0,T), \end{aligned} \tag{2.2.1}$$

$\Gamma_d = \Gamma_0 \cup \Gamma_b$ and $\Sigma_T^d = \Gamma_d \times (0,T)$. In order to rewrite system (2.1.1) in the cylindrical domain Q_T for all $t \in (0,T)$ consider the following diffeomorphism

$$\mathcal{T}_{\eta(t)} : \begin{cases} \Omega_{\eta(t)} & \longrightarrow \Omega_0, \\ (x,y) & \longmapsto (x,z) = \left(x, \frac{1+\eta^0(x)}{1+\eta(x,t)} y\right). \end{cases} \tag{2.2.2}$$

The variable z can be written under the form $z = \dfrac{y}{1+\widetilde{\eta}}$ with $\widetilde{\eta} = \frac{\eta - \eta^0}{1+\eta^0}$. We introduce the new unknowns

$$\widehat{\mathbf{u}}(x,z,t) = \mathbf{u}(\mathcal{T}_{\eta(t)}^{-1}(x,z),t) \quad \text{and} \quad \widehat{p}(x,z,t) = p(\mathcal{T}_{\eta(t)}^{-1}(x,z),t),$$

and we set $\widehat{\mathbf{u}}^0(x,z) = \mathbf{u}^0(\mathcal{T}_{\eta_1^0}^{-1}(x,z))$. With this change of variables,

$$p(x, 1+\eta(x,t), t) = \widehat{p}(x, 1+\eta^0(x,t), t) \quad \text{and} \quad \widehat{\mathbf{u}}(x, 1+\eta(x,t), t) = \mathbf{u}(x, 1+\eta^0(x,t), t),$$

for all $(x,t) \in (0,L) \times (0,T)$. The system satisfied by $(\widehat{\mathbf{u}}, \widehat{p}, \eta)$ is

$$\begin{aligned} &\widehat{\mathbf{u}}_t - \nu\Delta\widehat{\mathbf{u}} + \nabla\widehat{p} = \mathbf{G}(\widehat{\mathbf{u}}, \widehat{p}, \eta), \ \ \text{div}\, \widehat{\mathbf{u}} = \text{div}\, \mathbf{w}(\widehat{\mathbf{u}}, \eta) \ \text{ in } Q_T, \\ &\widehat{\mathbf{u}} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma_T^0, \\ &\widehat{u}_2 = 0 \ \text{ and } \ \widehat{p} + (1/2)|\widehat{\mathbf{u}}|^2 = 0 \ \text{ on } \Sigma_T^{i,o}, \\ &\widehat{\mathbf{u}} = 0 \ \text{ on } \Sigma_T^b, \ \ \widehat{\mathbf{u}}(0) = \widehat{\mathbf{u}}^0 \ \text{ on } \Omega_0, \\ &\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = \widehat{p} + \Psi(\widehat{\mathbf{u}}, \eta) \ \text{ on } \Sigma_T^0, \\ &\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0, L\} \times (0,T), \\ &\eta(0) = \eta_1^0 \ \text{ and } \ \eta_t(0) = \eta_2^0 \ \text{ in } \Gamma_s, \end{aligned} \tag{2.2.3}$$

with

$$\begin{aligned}
\mathbf{G}(\widehat{\mathbf{u}}, \widehat{p}, \eta) = & -\widetilde{\eta}\widehat{\mathbf{u}}_t + \left[z\widetilde{\eta}_t + \nu z \left(\frac{\widetilde{\eta}_x^2}{1+\widetilde{\eta}} - \widetilde{\eta}_{xx} \right) \right] \widehat{\mathbf{u}}_z \\
& + \nu \left[-2z\widetilde{\eta}_x \widehat{\mathbf{u}}_{xz} + \widetilde{\eta}\widehat{\mathbf{u}}_{xx} + \frac{z^2\widetilde{\eta}_x^2 - \widetilde{\eta}}{1+\widetilde{\eta}} \widehat{\mathbf{u}}_{zz} \right] \\
& + z(\widetilde{\eta}_x \widehat{p}_z - \widetilde{\eta}\widehat{p}_x)\mathbf{e}_1 - (1+\widetilde{\eta})\widehat{u}_1 \widehat{\mathbf{u}}_x + (z\widetilde{\eta}_x \widehat{u}_1 - \widehat{u}_2)\widehat{\mathbf{u}}_z,
\end{aligned}$$

$$\mathbf{w}[\widehat{\mathbf{u}}, \eta] = -\widetilde{\eta}\widehat{u}_1 \mathbf{e}_1 + z\widetilde{\eta}_x \widehat{u}_1 \mathbf{e}_2,$$

and

$$\Psi(\widehat{\mathbf{u}}, \eta) = \nu \left(\frac{\eta_x}{1+\widetilde{\eta}} \widehat{u}_{1,z} + \eta_x \widehat{u}_{2,x} - \frac{z\widetilde{\eta}_x \eta_x - 2}{1+\widetilde{\eta}} \widehat{u}_{2,z} \right).$$

In Section 2.4, in order to prove the existence of solution to system (2.2.6), derived from system (2.2.3) by a change of variables, we assume that η_1^0 is equal to η^0. In that special case, the function $\widetilde{\eta}$ is equal to 0 at time $t = 0$ which implies that $\mathcal{T}_{\eta(0)}$ is the identity. We also obtain that $\mathbf{w}(\widehat{\mathbf{u}}, \eta)$ is equal to 0 at time $t = 0$. But up to Section 2.4 and in the appendix, η^0 is chosen a priori, and not necessarily equal to η_1^0.

2.2.2 Final system and linearisation

In order to come back to a divergence free system consider the function $\overline{\mathbf{u}}$ defined by $\overline{\mathbf{u}} = \widehat{\mathbf{u}} - \mathbf{w}(\widehat{\mathbf{u}}, \eta)$. Set

$$M(\overline{\mathbf{u}}, \eta) = \begin{pmatrix} \dfrac{\overline{u}_1}{1+\widetilde{\eta}} \\ \dfrac{z\widetilde{\eta}_x}{1+\widetilde{\eta}} \overline{u}_1 + \overline{u}_2 \end{pmatrix} \quad \text{and} \quad N(\overline{\mathbf{u}}, \eta) = \begin{pmatrix} \dfrac{-\widetilde{\eta}\overline{u}_1}{1+\widetilde{\eta}} \\ \dfrac{z\widetilde{\eta}_x \overline{u}_1}{1+\widetilde{\eta}} \end{pmatrix}. \tag{2.2.4}$$

The function $\widehat{\mathbf{u}}$ can be expressed in terms of $\overline{\mathbf{u}}$ as follows

$$\widehat{\mathbf{u}} = M(\overline{\mathbf{u}}, \eta) = \overline{\mathbf{u}} + N(\overline{\mathbf{u}}, \eta).$$

To simplify the notation, we drop out the hat over p. Thus the system satisfied by $(\overline{\mathbf{u}}, \widehat{p}, \eta) = (\overline{\mathbf{u}}, p, \eta)$ is

$$\begin{aligned}
& \overline{\mathbf{u}}_t - \operatorname{div} \sigma(\overline{\mathbf{u}}, p) = \mathbf{F}(\overline{\mathbf{u}}, p, \eta), \ \operatorname{div} \overline{\mathbf{u}} = 0 \ \text{ in } Q_T, \\
& \overline{\mathbf{u}} = \eta_t \mathbf{e}_2 - \mathbf{w}(M(\overline{\mathbf{u}}, \eta), \eta) \ \text{ on } \Sigma_T^0, \\
& \overline{u}_2 = -w_2(M(\overline{\mathbf{u}}, \eta), \eta) \ \text{ and } \ p + (1/2)|\overline{\mathbf{u}} + \mathbf{w}(M(\overline{\mathbf{u}}, \eta), \eta)|^2 = 0 \ \text{ on } \Sigma_T^{i,o}, \\
& \overline{\mathbf{u}} = -\mathbf{w}(M(\overline{\mathbf{u}}, \eta), \eta) \ \text{ on } \Sigma_T^b, \ \ \overline{\mathbf{u}}(0) = \widehat{\mathbf{u}}^0 - \mathbf{w}(M(\overline{\mathbf{u}}, \eta), \eta)(0) \ \text{ in } \Omega_0, \\
& \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = p + \Psi(M(\overline{\mathbf{u}}, \eta), \eta) \ \text{ on } \Sigma_T^0, \\
& \eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0, L\} \times (0, T), \\
& \eta(0) = \eta_1^0 \ \text{ and } \ \eta_t(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{2.2.5}$$

with $\mathbf{F}(\overline{\mathbf{u}}, p, \eta) = \mathbf{G}(M(\overline{\mathbf{u}}, \eta), p, \eta) - \partial_t N(\overline{\mathbf{u}}, \eta) + \nu \Delta N(\overline{\mathbf{u}}, \eta)$.

Recall that $\mathbf{w}(\widehat{\mathbf{u}}, \eta) = -\widetilde{\eta}\widehat{u}_1\mathbf{e}_1 + z\widetilde{\eta}_x\widehat{u}_1\mathbf{e}_2$. Since $\widehat{u}_1 = 0$ on Σ_T^d and $\widetilde{\eta} = \widetilde{\eta}_x = 0$ on $\{0, L\} \times (0, T)$, we have $\mathbf{w}(\widehat{\mathbf{u}}, \eta) = 0$ on $\partial\Omega_0 \times (0, T)$. System (2.2.5) becomes

$$
\begin{aligned}
&\overline{\mathbf{u}}_t - \operatorname{div} \sigma(\overline{\mathbf{u}}, p) = \mathbf{F}(\overline{\mathbf{u}}, p, \eta), \ \operatorname{div} \overline{\mathbf{u}} = 0 \ \text{ in } Q_T, \\
&\overline{\mathbf{u}} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma_T^0, \\
&\overline{u}_2 = 0 \ \text{ and } \ p + (1/2)|\overline{\mathbf{u}}|^2 = 0 \ \text{ on } \Sigma_T^{i,o}, \\
&\overline{\mathbf{u}} = 0 \ \text{ on } \Sigma_T^b, \ \overline{\mathbf{u}}(0) = \overline{\mathbf{u}}^0 \ \text{ in } \Omega_0, \\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = p + H(\overline{\mathbf{u}}, \eta) \ \text{ on } \Sigma_T^0, \\
&\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0, L\} \times (0, T), \\
&\eta(0) = \eta_1^0 \ \text{ and } \ \eta_t(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{2.2.6}
$$

with $H(\overline{\mathbf{u}}, \eta) = \Psi(M(\overline{\mathbf{u}}, \eta), \eta)$ and $\overline{\mathbf{u}}^0 = \widehat{\mathbf{u}}^0 - \mathbf{w}(\widehat{\mathbf{u}}, \eta)(0)$.

In order to solve the system (2.2.6) with a fixed point argument, consider the following linear system

$$
\begin{aligned}
&\mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u}, p) = \mathbf{f}, \ \operatorname{div} \mathbf{u} = 0 \ \text{ in } Q_T, \\
&\mathbf{u} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma_T^0, \\
&u_2 = 0 \ \text{ and } \ p = \Theta \ \text{ on } \Sigma_T^{i,o}, \\
&\mathbf{u} = 0 \ \text{ on } \Sigma_T^b, \ \mathbf{u}(0) = \mathbf{u}^0 \ \text{ on } \Omega_0, \\
&\big[\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx}\big](x, t) \\
&\qquad = p(x, 1 + \eta^0(x, t), t) + h \ \text{ for } (x, t) \in (0, L) \times (0, T), \\
&\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0, L\} \times (0, T), \\
&\eta(0) = \eta_1^0 \ \text{ and } \ \eta_t(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{2.2.7}
$$

with $\mathbf{f} \in L^2(0, T; \mathbf{L}^2(\Omega_0))$, $h \in L^2(0, T; L^2(\Gamma_s))$ and $\Theta \in L^2(0, T; H^{1/2}(\Gamma_{i,o}))$.

2.3 Linear system

Recall that Ω_0 is given by (2.2.1) with a fixed $\eta^0 \in H^3(\Gamma_s) \cap H_0^2(\Gamma_s)$ such that $1 + \eta^0(x) > 0$ for all $x \in (0, L)$.

2.3.1 Function spaces

To deal with the mixed boundary condition for the Stokes system

$$
\begin{aligned}
&-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \ \ \operatorname{div} \mathbf{u} = 0 \ \text{ in } \Omega_0, \\
&\mathbf{u} = 0 \ \text{ on } \Gamma_d, \ u_2 = 0 \ \text{ and } p = 0 \ \text{ on } \Gamma_{i,o},
\end{aligned} \tag{2.3.1}
$$

introduce the space

$$\mathbf{V}^0_{n,\Gamma_d}(\Omega_0) = \{\mathbf{v} \in \mathbf{L}^2(\Omega_0) \mid \text{div } \mathbf{v} = 0 \text{ in } \Omega_0, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_d\},$$

and the orthogonal decomposition of $\mathbf{L}^2(\Omega_0) = L^2(\Omega_0, \mathbb{R}^2)$

$$\mathbf{L}^2(\Omega_0) = \mathbf{V}^0_{n,\Gamma_d}(\Omega_0) \oplus \text{grad } H^1_{\Gamma_{i,o}}(\Omega_0),$$

where $H^1_{\Gamma_{i,o}}(\Omega_0) = \{u \in H^1(\Omega_0) \mid u = 0 \text{ on } \Gamma_{i,o}\}$. Let $\Pi : \mathbf{L}^2(\Omega_0) \to \mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$ be the so-called Leray projector associated with this decomposition. If $\mathbf{u}$ belongs to $\mathbf{L}^2(\Omega_0)$ then $\Pi\mathbf{u} = \mathbf{u} - \nabla p_{\mathbf{u}} - \nabla q_{\mathbf{u}}$ where $p_{\mathbf{u}}$ and $q_{\mathbf{u}}$ are solutions to the following elliptic equations

$$\begin{aligned}&p_{\mathbf{u}} \in H^1_0(\Omega_0),\ \ \Delta p_{\mathbf{u}} = \text{div } \mathbf{u} \in H^{-1}(\Omega_0),\\ &q_{\mathbf{u}} \in H^1_{\Gamma_{i,o}}(\Omega_0),\ \ \Delta q_{\mathbf{u}} = 0,\ \ \frac{\partial q_{\mathbf{u}}}{\partial \mathbf{n}} = (\mathbf{u} - \nabla p_{\mathbf{u}})\cdot\mathbf{n} \text{ on } \Gamma_d,\ \ q_{\mathbf{u}} = 0 \text{ on } \Gamma_{i,o}.\end{aligned} \tag{2.3.2}$$

Through this paper the functions with vector values are written with a bold typography. For example $\mathbf{H}^2(\Omega_0) = H^2(\Omega_0, \mathbb{R}^2)$. As the boundary Γ_0 is not $\mathcal{C}^{2,1}$ it is not clear that the operator Π preserves the $\mathbf{H}^2$-regularity. However, with extra conditions on $\mathbf{u}$, this can be proved. Using the notations in [38, Theorem 11.7] we introduce the space $H^{3/2}_{00}(\Gamma_0) = [H^1_0(\Gamma_0), H^2_0(\Gamma_0)]_{1/2}$. The following lemma is proved in the appendix.

Lemma 2.3.1. Let $\mathbf{u}$ be in $\mathbf{H}^2(\Omega_0)$ satisfying div $\mathbf{u} = 0$, $\mathbf{u} = 0$ on Γ_b and $\mathbf{u} = g\mathbf{e}_2$ on Γ_0 with $g \in H^{3/2}_{00}(\Gamma_0)$. Then $\Pi\mathbf{u}$ belongs to $\mathbf{H}^2(\Omega_0)$.

The energy space associated with (2.3.1) is

$$V = \{\mathbf{u} \in \mathbf{H}^1(\Omega_0) \mid \text{div } \mathbf{u} = 0 \text{ in } \Omega_0,\ \mathbf{u} = 0 \text{ on } \Gamma_d,\ u_2 = 0 \text{ on } \Gamma_{i,o}\}.$$

The regularity result for (2.3.1) stated in Theorem 2.5.4 in the appendix allows us to introduce the Stokes operator A defined in $\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$ by

$$\mathcal{D}(A) = \mathbf{H}^2(\Omega_0) \cap V,$$

and for all $\mathbf{u} \in \mathcal{D}(A)$, $A\mathbf{u} = \nu\Pi\Delta\mathbf{u}$. We also use the notations

$$\mathbf{V}^s(\Omega_0) = \{\mathbf{u} \in \mathbf{H}^s(\Omega_0) \mid \text{div } \mathbf{u} = 0\},\ \ \mathbf{V}^s_{n,\Gamma_d}(\Omega_0) = \mathbf{V}^0_{n,\Gamma_d}(\Omega_0) \cap \mathbf{H}^s(\Omega_0),$$

for $s \geq 0$. For the Dirichlet boundary condition on Γ_0 set

$$\begin{aligned}\mathcal{L}^2(\Gamma_0) &= \{0\} \times L^2(\Gamma_0), & \mathcal{H}^{3/2}_{00}(\Gamma_0) &= \{0\} \times H^{3/2}_{00}(\Gamma_0),\\ \mathcal{H}^s(\Gamma_0) &= \{0\} \times H^s(\Gamma_0), & \mathcal{H}^s_0(\Gamma_0) &= \{0\} \times H^s_0(\Gamma_0) \text{ for } s \geq 0.\end{aligned}$$

For $s \geq 0$, the dual space of $\mathcal{H}^s(\Gamma_0)$ with $\mathcal{L}^2(\Gamma_0)$ as pivot space is denoted by $(\mathcal{H}^s(\Gamma_0))'$. Let $D \in \mathcal{L}(\mathcal{H}^{3/2}_{00}(\Gamma_0), \mathbf{H}^2(\Omega_0))$ be the operator defined by $D\mathbf{g} = \mathbf{w}$ where $(\mathbf{w}, p)$ is the solution to

$$\begin{aligned}&-\nu\Delta\mathbf{w} + \nabla p = 0,\ \ \text{div } \mathbf{w} = 0 \text{ in } \Omega_0,\\ &\mathbf{w} = \mathbf{g} \text{ on } \Gamma_0,\ \ \mathbf{w} = 0 \text{ on } \Gamma_b,\ \ w_2 = 0 \text{ and } p = 0 \text{ on } \Gamma_{i,o}.\end{aligned}$$

given by Theorem 2.5.4. Using a weak regularity result (Theorem 2.5.7) and interpolation techniques (see [38, Theorem 12.6] and [38, Remark 12.6]), D can be extended as a bounded linear operator in $\mathcal{L}(\mathcal{L}^2(\Gamma_0), \mathbf{H}^{1/2}(\Omega_0))$.

For space-time dependent functions we use the notations introduced in [39]:

$$\mathbf{L}^2(Q_T) = L^2(0,T;\mathbf{L}^2(\Omega_0)),\ \ \mathbf{H}^{p,q}(Q_T) = L^2(0,T;\mathbf{H}^p(\Omega_0)) \cap H^q(0,T;\mathbf{L}^2(\Omega_0)),\ p,q \geq 0,$$
$$L^2(\Sigma_T^s) = L^2(0,T;L^2(\Gamma_s)),\ \ H^{p,q}(\Sigma_T^s) = L^2(0,T;H^p(\Gamma_s)) \cap H^q(0,T;L^2(\Gamma_s)),\ \ p,q \geq 0.$$

2.3.2 Semigroup formulation of the linear system

We want to prove existence and regularity results for the coupled linear system (2.2.7). Let $\mathcal{R} \in \mathcal{L}(H^{1/2}(\Gamma_{i,o}), H^1(\Omega))$ be a lifting operator. Classically we transform (2.2.7) into a system with homogeneous boundary conditions (for the pressure) by looking for a solution to (2.2.7) under the form $(\mathbf{u}, p, \eta) = (\mathbf{u}, p_1, \eta) + (0, \mathcal{R}(\Theta), 0)$ with $(\mathbf{u}, p_1, \eta)$ solution to

$$(2.3.3)\qquad \begin{aligned}
&\mathbf{u}_t - \mathrm{div}\ \sigma(\mathbf{u}, p_1) = \mathbf{f} - \nabla\mathcal{R}(\Theta),\ \ \mathrm{div}\ \mathbf{u} = 0\ \text{ in } Q_T,\\
&\mathbf{u} = \eta_t \mathbf{e}_2\ \text{ for } \Sigma_T^0,\\
&u_2 = 0\ \text{ and }\ p_1 = 0\ \text{ on } \Sigma_T^{i,o},\\
&\mathbf{u} = 0\ \text{ on } \Sigma_T^b,\ \ \mathbf{u}(0) = \mathbf{u}^0\ \text{ in } \Omega_0,\\
&\big[\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx}\big](x,t)\\
&\qquad = \Big[p_1 + \mathcal{R}(\Theta)\Big](x, 1+\eta^0(x,t), t) + h\ \text{ for } (x,t) \in (0,L) \times (0,T),\\
&\eta = 0\ \text{ and }\ \eta_x = 0\ \text{ on } \{0, L\} \times (0,T),\\
&\eta(0) = \eta_1^0\ \text{ and }\ \eta_t(0) = \eta_2^0\ \text{ in } \Gamma_s.
\end{aligned}$$

Set $F = \mathbf{f} - \nabla\mathcal{R}(\Theta)$. As the boundary Γ_0 may not be flat and the beam equation is written on Γ_s consider the transport operator $\mathcal{U} \in \mathcal{L}(L^2(\Gamma_0), L^2(\Gamma_s))$ defined by

$$(\mathcal{U}g)(x,1) = g(x, 1+\eta^0(x)) \quad \text{for all } g \in L^2(\Gamma_0).$$

We can easily verify that $\mathcal{U}$ is an isomorphism from $L^2(\Gamma_0)$ to $L^2(\Gamma_s)$, and that

$$(\mathcal{U}^{-1}\widetilde{g})(x, 1+\eta^0(x)) = \widetilde{g}(x,1) \quad \text{for all } \widetilde{g} \in L^2(\Gamma_s).$$

Moreover $\mathcal{U}^{-1} = \mathcal{U}^*$, if $L^2(\Gamma_0)$ and $L^2(\Gamma_s)$ are equipped with the inner products

$$(f,g)_{L^2(\Gamma_0)} = \big(f(\cdot, 1+\eta^0(\cdot))g(\cdot, 1+\eta^0(\cdot))\big)_{L^2(0,L)},$$
and
$$\Big(\widetilde{f}, \widetilde{g}\Big)_{L^2(\Gamma_s)} = \Big(\widetilde{f}(\cdot,1)\widetilde{g}(\cdot,1)\Big)_{L^2(0,L)}.$$

In order to express the pressure p_1 in terms of $\Pi\mathbf{u}$ and η we introduce the Neumann-to-Dirichlet operator $N_s \in \mathcal{L}(L^2(\Gamma_0))$ defined by $N_{s,0}(g) = \pi|_{\Gamma_0}$ where $g \in L^2(\Gamma_0)$ and π is the solution to

$$\begin{cases} \Delta\pi = 0 \text{ in } \Omega_0, \\ \pi = 0 \text{ on } \Gamma_{i,o}, \ \dfrac{\partial\pi}{\partial\mathbf{n}} = g(1+(\eta^0)^2)^{-1/2} \text{ on } \Gamma_0 \text{ and } \dfrac{\partial\pi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_b. \end{cases}$$

As in [56, Lemma 3.1], $N_{s,0}$ is a non-negative symmetric and compact operator in $L^2(\Gamma_0)$. Hence, as $\mathcal{U} \in \mathrm{isom}(L^2(\Gamma_0), L^2(\Gamma_s))$ and $\mathcal{U}^{-1} = \mathcal{U}^*$, the operator $N_s = \mathcal{U}N_{s,0}\mathcal{U}^{-1}$ is a non-negative symmetric and compact operator in $L^2(\Gamma_s)$. Consequently, the operator $(I_{L^2(\Gamma_s)} + N_s)$ is an automorphism in $L^2(\Gamma_s)$.

We also define the operator $N_0 \in \mathcal{L}(H^{-1/2}(\Gamma_d), L^2(\Gamma_s))$ by $N_0(v) = \mathcal{U}(\rho|_{\Gamma_0})$ for all $v \in H^{-1/2}(\Gamma_d)$, where ρ is the solution to

$$\begin{cases} \Delta\rho = 0 \text{ in } \Omega_0, \\ \rho = 0 \text{ on } \Gamma_{i,o} \text{ and } \dfrac{\partial\rho}{\partial\mathbf{n}} = v \text{ on } \Gamma_d. \end{cases}$$

Finally set $D_s(\eta_t) = D(\mathcal{U}^{-1}\eta_t\mathbf{e}_2)$. The following lemma is similar to [56, Lemma 3.1] and is a direct application of Theorem 2.5.10 in the appendix.

Lemma 2.3.2. A pair $(\mathbf{u}, p_1) \in \mathbf{H}^{2,1}(Q_T) \times L^2(0,T;H^1(\Omega_0))$ obeys the fluid equations of (2.3.3) if and only if

$$\begin{aligned} &\Pi\mathbf{u}' = A\Pi\mathbf{u} + (-A)\Pi D_s(\eta_t) + \Pi F, \ \ \mathbf{u}(0) = \mathbf{u}^0, \\ &(I-\Pi)\mathbf{u} = (I-\Pi)D_s(\eta_t), \ \ p_1 = \rho - q_t + p_F, \end{aligned}$$

where

- $q \in H^1(0,T;H^1(\Omega_0))$ is the solution to

$$\Delta q = 0 \text{ in } Q_T, \ \ \rho = 0 \text{ on } \Sigma_T^{i,o}, \ \ \frac{\partial q}{\partial\mathbf{n}} = \mathcal{U}^{-1}\eta_t\mathbf{e}_2\cdot\mathbf{n} \text{ on } \Sigma_T^0, \ \ \frac{\partial q}{\partial\mathbf{n}} = 0 \text{ on } \Sigma_T^b.$$

- $\rho \in L^2(0,T;H^1(\Omega_0))$ is the solution to

$$\Delta\rho = 0 \text{ in } Q_T, \ \ \rho = 0 \text{ on } \Sigma_T^{i,o}, \ \ \frac{\partial\rho}{\partial\mathbf{n}} = \nu\Delta\Pi\mathbf{u}\cdot\mathbf{n} \text{ on } \Sigma_T^d.$$

- $p_F \in L^2(0,T;H^1(\Omega_0))$ is given by the identity $(I-\Pi)F = \nabla p_F$.

Using Lemma 2.3.2 the pressure in the beam equation can be decomposed as follows $p_1 = \nu N_0(\Delta\Pi\mathbf{u}\cdot\mathbf{n}) - \partial_t N_s(\eta_t) + \mathcal{U}(p_F|_{\Gamma_0})$. Hence the beam equation becomes

$$(I_{L^2(\Gamma_s)} + N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = \nu N_0(\Delta\Pi\mathbf{u}\cdot\mathbf{n}) + \mathcal{U}\big[(p_F + \mathcal{R}(\Theta))|_{\Gamma_0}\big] + h.$$

The system (2.3.3) can be rewritten in terms of $(\Pi\mathbf{u}, \eta, \eta_t) = (\Pi\mathbf{u}, \eta_1, \eta_2)$ as

$$\text{(2.3.4)} \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix}\Pi\mathbf{u}\\ \eta_1\\ \eta_2\end{pmatrix} = \mathcal{A}\begin{pmatrix}\Pi\mathbf{u}\\ \eta_1\\ \eta_2\end{pmatrix} + \mathbf{F}, \\ (I-\Pi)\mathbf{u} = (I-\Pi)D_s(\eta_t), \end{cases}$$

where $\mathcal{A}$ is the unbounded operator in

$$\mathbf{H} = \mathbf{V}^0_{n,\Gamma_d}(\Omega_0) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s),$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{(\Pi\mathbf{u}, \eta_1, \eta_2) \in \mathbf{V}^2_{n,\Gamma_d}(\Omega_0)\times(H^4(\Gamma_s)\cap H^2_0(\Gamma_s))\times H^2_0(\Gamma_s) \mid \Pi\mathbf{u}-\Pi D_s(\eta_2) \in \mathcal{D}(A)\},$$

defined by

$$\text{(2.3.5)} \qquad \mathcal{A} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I+N_s)^{-1}\end{pmatrix}\begin{pmatrix} A & 0 & (-A)\Pi D_s \\ 0 & 0 & I \\ \nu N_0(\Delta(\cdot)\cdot\mathbf{n}) & \beta\Delta_s - \alpha\Delta_s^2 & \delta\Delta_s\end{pmatrix},$$

with $\Delta_s = \partial_{xx}$ and

$$\mathbf{F} = \begin{pmatrix} \Pi F \\ 0 \\ (I+N_s)^{-1}(\mathcal{U}[(p_F + \mathcal{R}(\Theta))|_{\Gamma_0}] + h)\end{pmatrix}.$$

2.3.3 Analyticity of $\mathcal{A}$

Let $(A_{\alpha,\beta}, \mathcal{D}(A_{\alpha,\beta}))$ be the unbounded operator in $L^2(\Gamma_s)$ defined by $\mathcal{D}(A_{\alpha,\beta}) = H^4(\Gamma_s)\cap H^2_0(\Gamma_s)$ and for all $\eta \in \mathcal{D}(A_{\alpha,\beta})$, $A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha\eta_{xxxx}$. The operator $A_{\alpha,\beta}$ is self-adjoint and is an isomorphism from $\mathcal{D}(A_{\alpha,\beta})$ to $L^2(\Gamma_s)$. The space $\mathbf{H}$ will be equipped with the inner product

$$\langle(\mathbf{u}, \eta_1, \eta_2), (\mathbf{v}, \zeta_1, \zeta_2)\rangle_{\mathbf{H}} = \langle\mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)} + \langle\eta_1, \zeta_1\rangle_{H^2_0(\Gamma_s)} + \langle\eta_2, \zeta_2\rangle_{L^2(\Gamma_s)},$$

with $\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$ endowed with the natural scalar product of $L^2(\Omega_0)$ and

$$\langle\eta_1, \zeta_1\rangle_{H^2_0(\Gamma_s)} = \int_{\Gamma_s}(-A_{\alpha,\beta})^{1/2}\eta_1(-A_{\alpha,\beta})^{1/2}\zeta_1 = \int_{\Gamma_s}(\beta\eta_{1,x}\zeta_{1,x} + \alpha\eta_{1,xx}\zeta_{1,xx}).$$

This scalar product on $H^2_0(\Omega_0)$ is used to simplify calculations involving the operator $A_{\alpha,\beta}$. The unbounded operator relative to the beam $(A_s, \mathcal{D}(A_s))$ in

$$H_s = H^2_0(\Gamma_s) \times L^2(\Gamma_s),$$

is defined by $\mathcal{D}(A_s) = (H^4(\Gamma_s)\cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s)$ and $A_s = \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} & \delta\Delta_s\end{pmatrix}$.

Theorem 2.3.1. The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$.

Proof. The idea of the proof is to split the operator $\mathcal{A}$ in two parts. The principal part of $\mathcal{A}$ will be the infinitesimal generator of an analytic semigroup on $\mathbf{H}$ and the rest will be a perturbation bounded with respect to the principal part. Set $K_s = (I + N_s)^{-1} - I$. The operator $\mathcal{A}$ can be written

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2,$$

with

$$\mathcal{A}_1 = \begin{pmatrix} A & 0 & (-A)\Pi D_s \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix},$$

and

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nu(I + N_s)^{-1} N_0(\Delta(\cdot) \cdot \mathbf{n}) & K_s A_{\alpha,\beta} & \delta K_s \Delta_s \end{pmatrix}.$$

According to [52, Section 3.2, Theorem 2.1], the result is a direct consequence of Theorem 2.3.2 and 2.3.3. □

Theorem 2.3.2. The operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$.

Proof. The proof follows the techniques used in [56, Theorem 3.5]. The first part is to prove that the semigroup $\mathcal{A}_1$ is strongly continuous. This property, established using regularity results on the unsteady Stokes equations, is proved in the appendix (Lemma 2.5.3).

The next step is to estimate the resolvent of $\mathcal{A}_1$. Using a perturbation argument to ensure the existence of the resolvent, we have, at least for $\mathrm{Re}(\lambda) > 0$,

$$(\lambda I - \mathcal{A}_1)^{-1} = \begin{pmatrix} (\lambda I - A)^{-1} & (0 \ (\lambda I - A)^{-1}(-A)\Pi D_s)(\lambda I - A_s)^{-1} \\ 0 & (\lambda I - A_s)^{-1} \end{pmatrix},$$

where $(\lambda I - A_s)^{-1}$ is given by

$$(\lambda I - A_s)^{-1} = \begin{pmatrix} \mathcal{V}^{-1}(\lambda I - \delta\Delta_s) & \mathcal{V}^{-1} \\ \mathcal{V}^{-1} A_{\alpha,\beta} & \lambda\mathcal{V}^{-1} \end{pmatrix},$$

with $\mathcal{V} = \lambda^2 I - \lambda\delta\Delta_s - A_{\alpha,\beta}$. From [16] we know that there exists $a \in \mathbb{R}$ and $\frac{\pi}{2} < \theta_0 < \pi$ such that for all λ in $S_{a,\theta_0} = \{\lambda \in \mathbb{C} \mid \lambda \neq a, \ |\arg(\lambda - a)| < \theta_0\}$

$$\left\| (\lambda I - A_s)^{-1} \right\|_{\mathcal{L}(H_s)} \leq \frac{C_s}{|\lambda - a|}.$$

The Stokes operator A is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$ and the proof of Theorem 2.5.5 gives the existence of $\frac{\pi}{2} < \theta_1 < \pi$ such that for all $\lambda \in S_{0,\theta_1}$

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(\mathbf{V}^0_{n,\Gamma_d}(\Omega_0))} \le \frac{C_A}{|\lambda|}.$$

Choose $a' > a$ and $\theta' = \min(\theta_0, \theta_1)$. For all $\lambda \in S_{a',\theta'}$ and all $(\mathbf{f}, \Phi) \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_0) \times H_s$ we have

$$(\lambda I - \mathcal{A}_1)^{-1} \begin{pmatrix} \mathbf{f} \\ \Phi \end{pmatrix} = \begin{pmatrix} (\lambda I - A)^{-1}\mathbf{f} + (\lambda I - A)^{-1}(-A)\Pi D_s((\lambda I - A_s)^{-1}\Phi)_2 \\ (\lambda I - A_s)^{-1}\Phi \end{pmatrix}.$$

Remark that $(\lambda I - A)^{-1}(-A)\Pi D_s = \Pi D_s - \lambda(\lambda I - A)^{-1}\Pi D_s$. Using the previous estimates for the resolvent of A and A_s and the continuity of the operator ΠD_s we obtain

$$\begin{aligned}
&\left\|(\lambda I - \mathcal{A}_1)^{-1} \begin{pmatrix} \mathbf{f} \\ \Phi \end{pmatrix}\right\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)} \\
&\le \frac{C_A}{|\lambda - a'|} \|\mathbf{f}\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)} + \frac{C_{\Pi D_s} C_s}{|\lambda - a'|} \|\Phi\|_{H_s} + \frac{C_A C_{\Pi D_s} C_s}{|\lambda - a'|} \|\Phi\|_{H_s} + \frac{C_s}{|\lambda - a'|} \|\Phi\|_{H_s}.
\end{aligned}$$

Hence there exists a constant $C > 0$ such that

$$\left\|(\lambda I - \mathcal{A}_1)^{-1}\right\|_{\mathcal{L}(\mathbf{H})} \le \frac{C}{|\lambda - a'|},$$

for all $\lambda \in S_{a',\theta'}$ and $\mathcal{A}_1$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$. $\square$

Theorem 2.3.3. The operator $(\mathcal{A}_2, \mathcal{D}(\mathcal{A}_2) = \mathcal{D}(\mathcal{A}))$ is $\mathcal{A}_1$-bounded with relative bound zero.

Proof. We proceed as in [56]. Split the operator $\mathcal{A}_2$ in three parts $\mathcal{A}_2 = \mathcal{A}_{2,1} + \mathcal{A}_{2,2} + \mathcal{A}_{2,3}$ with

$$\mathcal{A}_{2,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nu(I + N_s)^{-1} N_0(\Delta(\cdot) \cdot \mathbf{n}) & 0 & 0 \end{pmatrix},$$

$$\mathcal{A}_{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & K_s A_{\alpha,\beta} & 0 \end{pmatrix}, \quad \mathcal{A}_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta K_s \Delta_s \end{pmatrix}.$$

The following lemma is an adaptation of [56, Proposition 3.3].

Lemma 2.3.3. The norm

$$(\Pi\mathbf{u},\eta_1,\eta_2)\mapsto \|(\Pi\mathbf{u},\eta_1,\eta_2)\|_{\mathbf{H}} + \|A\Pi\mathbf{u} + (-A)\Pi D_s\eta_2\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)} + \|A_s(\eta_1,\eta_2)\|_{H_s},$$

is a norm on $\mathcal{D}(\mathcal{A})$ which is equivalent to the norm

$$(\Pi\mathbf{u},\eta_1,\eta_2)\mapsto \|\Pi\mathbf{u}\|_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2_0(\Gamma_s)}.$$

For $\mathcal{A}_{2,2}$ and $\mathcal{A}_{2,3}$ we can use [56, Lemma 3.9] and [56, Lemma 3.10] to prove that there exists $0<\theta_1<1$ and $0<\theta_2<1$ such that $\mathcal{A}_{2,2}$ (respectively $\mathcal{A}_{2,3}$) is bounded from $\mathcal{D}((-\mathcal{A}_1)^{\theta_1})$ (respectively from $\mathcal{D}((-\mathcal{A}_1)^{\theta_2})$) into $\mathbf{H}$. Hence, according to [52, Section 3.2, Corollary 2.4], the operators $\mathcal{A}_{2,2}$ and $\mathcal{A}_{2,3}$ are $\mathcal{A}_1$-bounded with relative bound zero. It remains to prove that $\mathcal{A}_{2,1}$ is $\mathcal{A}_1$-bounded with relative bound zero.

Lemma 2.3.4. For all $\varepsilon>0$ there exists a constant C_ε such that

$$\|N_0(\Delta\mathbf{u}\cdot\mathbf{n})\|_{L^2(\Gamma_s)} \le \varepsilon\,\|\mathbf{u}\|_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)} + C_\varepsilon\,\|\mathbf{u}\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)}, \tag{2.3.6}$$

for all $\mathbf{u}\in\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)$.

Proof. Using the transposition method, a density argument (as in Theorem 2.5.6 and Theorem 2.5.7) and interpolation, the operator N_0 can be defined as a continuous operator from $H^{-1}(\Gamma_s)$ into $L^2(\Gamma_s)$. We prove the lemma by contradiction. Assume that there exists a sequence $\mathbf{u}_k\in\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)$ such that

$$\|N_0(\Delta\mathbf{u}_k\cdot\mathbf{n})\|_{L^2(\Gamma_s)}=1,\ \|\mathbf{u}_k\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)}\longrightarrow 0,\ \|\mathbf{u}_k\|_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)}\le M,$$

with $M>0$ a fixed constant. By reflexivity of the space $\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)$, up to a subsequence, there exists $\mathbf{u}\in\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)$ such that $\mathbf{u}_k\rightharpoonup\mathbf{u}$ in $\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)$. Since $\|\mathbf{u}_k\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)}\longrightarrow 0$, we obtain $\mathbf{u}=0$. Then $\Delta\mathbf{u}_k\cdot\mathbf{n}\rightharpoonup 0$ in $H^{-1/2}(\Gamma_s)$ and the compact embedding of $H^{-1/2}(\Gamma_s)$ into $H^{-1}(\Gamma_s)$ ensures that $\Delta\mathbf{u}_k\cdot\mathbf{n}\longrightarrow 0$ in $H^{-1}(\Gamma_s)$. Finally the continuity of N_0 implies that $N_0(\Delta\mathbf{u}_k\cdot\mathbf{n})\longrightarrow 0$ in $L^2(\Gamma_s)$ which contradicts $\|N_0(\Delta\mathbf{u}_k\cdot\mathbf{n})\|_{L^2(\Gamma_s)}=1$. □

We come back to the proof that $\mathcal{A}_{2,1}$ is $\mathcal{A}_1$-bounded. Using the estimate (2.3.6) and the norm equivalence of Lemma 2.3.3 it follows that for all $(\Pi\mathbf{u},\eta_1,\eta_2)\in\mathcal{D}(\mathcal{A}_1)$

$$\begin{aligned}
\left\|\mathcal{A}_{2,1}\begin{pmatrix}\Pi\mathbf{u}\\ \eta_1\\ \eta_2\end{pmatrix}\right\|_{\mathbf{H}} &\le \varepsilon\,\|\Pi\mathbf{u}\|_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)} + C_\varepsilon\,\|\Pi\mathbf{u}\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)}\\
&\le \varepsilon(\|\Pi\mathbf{u}\|_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_0)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2_0(\Gamma_s)}) + C_\varepsilon\,\|\Pi\mathbf{u}\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)}\\
&\le C_1\varepsilon\left\|\mathcal{A}_1\begin{pmatrix}\Pi\mathbf{u}\\ \eta_1\\ \eta_2\end{pmatrix}\right\|_{\mathbf{H}} + C_{2,\varepsilon}\left\|\begin{pmatrix}\Pi\mathbf{u}\\ \eta_1\\ \eta_2\end{pmatrix}\right\|_{\mathbf{H}}.
\end{aligned}$$

This concludes the proof that $\mathcal{A}_2$ is $\mathcal{A}_1$-bounded with relative bound zero. □

2.3.4 Regularity results

We have seen that the system (2.3.3) can be rewritten

$$\tag{2.3.7} \begin{cases} \dfrac{d}{dt}\begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathbf{F}, \quad \begin{pmatrix} \Pi\mathbf{u}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} \Pi\mathbf{u}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\ (I-\Pi)\mathbf{u} = (I-\Pi)D_s(\eta_2). \end{cases}$$

We remark that there is no condition on $(I-\Pi)\mathbf{u}^0$. As in [56], in order to satisfy the equality $(I-\Pi)\mathbf{u} = (I-\Pi)D_s(\eta_2)$ at time $t=0$, we introduce a subspace of initial conditions belonging to $\mathbf{V}^1(\Omega_0)\times H_s$ and satisfying a compatibility condition

$$\mathbf{H}_{\mathrm{cc}} = \{(\mathbf{u}^0,\eta_1^0,\eta_2^0) \in \mathbf{V}^1(\Omega_0)\times H_s \mid (I-\Pi)\mathbf{u}^0 = (I-\Pi)D_s(\eta_2^0)\}.$$

To obtain maximal regularity results, introduce the space $[\mathcal{D}(\mathcal{A}),\mathbf{H}]_{1/2}$ given by

$$\begin{aligned}[\mathcal{D}(\mathcal{A}),\mathbf{H}]_{1/2} = \{(\Pi\mathbf{u},\eta_1,\eta_2) \in \mathbf{V}^1_{n,\Gamma_d}(\Omega_0)\times(H^3(\Gamma_s)\cap H_0^2(\Gamma_s))\times H_0^1(\Gamma_s) \mid \\ \Pi\mathbf{u}-\Pi D_s(\eta_2)\in V\}.\end{aligned}$$

It is equipped with the norm

$$(\Pi\mathbf{u},\eta_1,\eta_2)\longmapsto\left(\|\Pi\mathbf{u}\|^2_{\mathbf{H}^1(\Omega_0)}+\|\eta_1\|^2_{H^3(\Gamma_s)}+\|\eta_2\|^2_{H^1(\Gamma_s)}\right)^{1/2}.$$

If the initial condition $(\Pi\mathbf{u}^0,\eta_1^0,\eta_2^0)$ belongs to $[\mathcal{D}(\mathcal{A}),\mathbf{H}]_{1/2}$, and if the compatibility condition $(I-\Pi)\mathbf{u}^0 = (I-\Pi)D_s(\eta_2^0)$ is satisfied, then $(\mathbf{u}^0,\eta_1^0,\eta_2^0)$ belongs to

$$\mathcal{X}(\Omega_0) = \{(\mathbf{u}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}_{\mathrm{cc}} \mid (\Pi\mathbf{u}^0,\eta_1^0,\eta_2^0)\in[\mathcal{D}(\mathcal{A}),\mathbf{H}]_{1/2}\}.$$

The space $\mathcal{X}(\Omega_0)$ is equipped with the norm

$$\mathbf{x}^0 = (\mathbf{u}^0,\eta_1^0,\eta_2^0)\longmapsto\left\|\mathbf{x}^0\right\|_{\mathcal{X}(\Omega_0)} = \left(\left\|\Pi\mathbf{u}^0\right\|^2_{\mathbf{H}^1(\Omega_0)}+\left\|\eta_1^0\right\|^2_{H^3(\Gamma_s)}+\left\|\eta_2^0\right\|^2_{H^1(\Gamma_s)}\right)^{1/2}.$$

We notice that the above mapping is indeed a norm since $(I-\Pi)\mathbf{u}^0 = (I-\Pi)D_s(\eta_2^0)$ if $\mathbf{x}^0 = (\mathbf{u}^0,\eta_1^0,\eta_2^0)\in\mathcal{X}(\Omega_0)$. Defining W_T by

$$W_T = \mathbf{L}^2(Q_T)\times L^2(0,T;H^{1/2}(\Gamma_{i,o}))\times L^2(0,T;L^2(\Gamma_s)),$$

we obtain the main theorem of this section.

Theorem 2.3.4. For all $(\mathbf{u}^0,\eta_1^0,\eta_2^0)$ in $\mathcal{X}(\Omega_0)$ and $(\mathbf{f},\Theta,h)$ in W_T, system (2.2.7) admits a unique solution $(\mathbf{u},p,\eta)\in\mathbf{H}^{2,1}(Q_T)\times L^2(0,T;H^1(\Omega_0))\times H^{4,2}(\Sigma_T^s)$. This solution satisfies

$$\tag{2.3.8} \begin{aligned} &\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)}+\|\eta\|_{H^{4,2}(\Sigma_T^s)}+\|p\|_{L^2(0,T;H^1(\Omega_0))} \\ &\le C_L(\left\|(\mathbf{u}^0,\eta_1^0,\eta_2^0)\right\|_{\mathcal{X}(\Omega_0)}+\|(\mathbf{f},\Theta,h)\|_{W_T}).\end{aligned}$$

Proof. According to [10, Part II, Section 1.3, Theorem 3.1] there exists a unique solution $(\mathbf{u}, \eta_1, \eta_2)$ to (2.3.7) and the following estimate holds

$$\begin{aligned}
&\|\Pi\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)} + \|(\eta_1,\eta_2)\|_{L^2(0,T;\mathcal{D}(A_s))\cap H^1(0,T;H_s)} \\
&\qquad \le C\Big(\left\|(\Pi\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{[\mathcal{D}(\mathcal{A}),\mathbf{H}]_{1/2}} + \|\mathbf{F}\|_{L^2(0,T;\mathbf{H})}\Big) \\
&\|(I-\Pi)\mathbf{u}\|_{L^2(0,T;\mathbf{H}^2(\Omega_0))} + \|(I-\Pi)\mathbf{u}\|_{H^1(0,T;\mathbf{H}^{1/2}(\Omega_0))} \\
&\qquad \le C\,\|(\eta_1,\eta_2)\|_{L^2(0,T;\mathcal{D}(A_s))\cap H^1(0,T;H_s)},
\end{aligned}$$

with the estimate on $(I-\Pi)\mathbf{u}$ coming from the properties of the operator D_s and the identity $(I-\Pi)\mathbf{u} = (I-\Pi)\eta_2$. Estimate (2.3.8) follows by writing $\mathbf{F}$ explicitly. $\square$

Remark 2.3.1. Let T_0 be a fixed time with $T < T_0$. The constant C_L in the previous statement can be chosen independent of T for all $T < T_0$. If we extend all the nonhomogeneous terms on $[T, T_0]$ by 0 (still denoted by $(\mathbf{f}, \Theta, h)$) the previous result implies that there exists a unique solution $(\mathring{\mathbf{u}}, \mathring{p}, \mathring{\eta})$ of (2.2.7) and the following estimate holds

$$\begin{aligned}
&\left\|\mathring{\mathbf{u}}\right\|_{\mathbf{H}^{2,1}(Q_{T_0})} + \left\|\mathring{\eta}\right\|_{H^{4,2}(\Sigma^s_{T_0})} + \left\|\mathring{p}\right\|_{L^2(0,T_0;H^1(\Omega_0))} \\
&\le C_L(T_0)\Big(\left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)} + \|(\mathbf{f},\Theta,h)\|_{W_{T_0}}\Big).
\end{aligned}$$

The uniqueness yields $(\mathbf{u}, p, \eta) = (\mathring{\mathbf{u}}, \mathring{p}, \mathring{\eta})$ on $[0, T]$ and the constant C_L in Theorem 2.3.4 can be taken as $C_L = C_L(T_0)$.

2.4 Nonlinear coupled system

Throughout this section, excepted for Theorem 2.4.4 which is stated in a rectangular domain, Ω_0 is given by (2.2.1) for any fixed $\eta^0 \in H^3(\Gamma_s)\cap H_0^2(\Gamma_s)$ such that $1+\eta^0(x) > 0$ for all $x \in (0, L)$. We prove the existence of strong solutions for the complete nonlinear system (2.2.6). Let $T_0 > 0$ be a given time, fixed for this section. Let $\tilde{\mathcal{X}}(\Omega_0)$ be the affine subspace of $\mathcal{X}(\Omega_0)$ defined by

$$\tilde{\mathcal{X}}(\Omega_0) = \{(\mathbf{u}^0, \eta_1^0, \eta_2^0) \in \mathcal{X}(\Omega_0) \mid \eta_1^0 = \eta^0\},$$

that is, the space where the initial data of the beam η_0^1 and the geometric η^0 are equal. For $T > 0$, set

$$\begin{aligned}
\mathcal{Y}_T = \{&(\overline{\mathbf{u}}, p, \eta) \in \mathbf{H}^{2,1}(Q_T) \times L^2(0,T;H^1(\Omega_0)) \times H^{4,2}(\Sigma^s_T) \mid \\
&\overline{\mathbf{u}} = 0 \text{ on } \Sigma^b_T,\ \overline{\mathbf{u}} = \eta_t\mathbf{e}_2 \text{ on } \Sigma^0_T,\ \overline{u}_2 = 0 \text{ on } \Sigma^{i,o}_T,\ (\overline{\mathbf{u}}(0), \eta(0), \eta_t(0)) \in \tilde{\mathcal{X}}(\Omega_0)\}.
\end{aligned}$$

The usual norm on $\mathbf{H}^{2,1}(Q_T) \times L^2(0,T;H^1(\Omega_0)) \times H^{4,2}(\Sigma^s_T)$ is denoted by $\|\cdot\|_{\mathcal{Y}_T}$.

2.4.1 Estimates

For every $\mathbf{x}^0 = (\mathbf{u}^0, \eta_1^0, \eta_2^0) \in \tilde{\mathcal{X}}(\Omega_0)$, $R > 0$, $\mu > 0$ and $T > 0$, define the ball

$$\mathcal{B}(\mathbf{x}^0, R, \mu, T) = \{(\overline{\mathbf{u}}, p, \eta) \in \mathcal{Y}_T \mid (\overline{\mathbf{u}}(0), \eta(0), \eta_t(0)) = \mathbf{x}^0,\ \|(\overline{\mathbf{u}}, p, \eta)\|_{\mathcal{Y}_T} \leq R,$$
$$\|(1+\eta)^{-1}\|_{L^\infty(\Sigma_T^s)} \leq 2\mu\}.$$

For a given $\mathbf{x}^0 = (\mathbf{u}^0, \eta_1^0, \eta_2^0) \in \tilde{\mathcal{X}}(\Omega_0)$, Theorem 2.3.4 ensures the existence of $R > 0$ and $\mu > 0$ such that $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$ is non empty for $T > 0$ small enough (such a triple (R, μ, T) is explicitly chosen in the beginning of the proof of Theorem 2.4.2).

Throughout this section, $C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)})$ denotes a constant, depending on T_0, $R > 0$, $\mu > 0$, $\|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}$ which may vary from one statement to another, but which is independent of T.

The following lemmas are used to estimate the nonlinear terms (see Theorem 2.4.1).

Lemma 2.4.1. There exists a constant C_0 depending on T_0 such that, for all $0 < T < T_0$ and all $u \in H^{2,1}(Q_T)$ satisfying $u(0) = 0$, the following estimate holds

$$\|u\|_{L^\infty(0,T;H^1(\Omega_0))} + \|u\|_{L^4(0,T;L^\infty(\Omega_0))} \leq C_0 \|u\|_{H^{2,1}(Q_T)}.$$

Moreover for all $v \in H^{4,2}(\Sigma_T^s)$ satisfying $v(0) = 0$ the following estimate holds

$$\|v\|_{L^\infty(0,T;H^3(\Gamma_s))} \leq C_0 \|v\|_{H^{4,2}(\Sigma_T^s)}.$$

If in addition $v_t(0) = 0$, then

$$\|v_t\|_{L^\infty(0,T;H^1(\Gamma_s))} + \|v_t\|_{L^2(0,T;H^2(\Gamma_s))} \leq C_0 \|v\|_{H^{4,2}(\Sigma_T^s)}.$$

Proof. These estimates come from interpolation results (see [39]). The only thing to prove is that the continuity constants can be made independent of T. Let $\overline{u}$ be the function defined by $\overline{u} = 0$ on $[T - T_0, 0]$ and $\overline{u} = u$ on $[0, T]$. As $u(0) = 0$, the function $\overline{u}$ is still in $H^{2,1}(Q_T)$ and, using interpolation estimates, we have

$$\|\overline{u}\|_{L^\infty(T-T_0,T;H^1(\Omega_0))} \leq C(T_0) \|\overline{u}\|_{L^2(T-T_0,T;H^2(\Omega_0)) \cap H^1(T-T_0,T;L^2(\Omega_0))}.$$

This implies that $\|u\|_{L^\infty(0,T;H^1(\Omega_0))} \leq C_0 \|u\|_{H^{2,1}(Q_T)}$ with $C_0 = C(T_0)$. The other estimates follow from the same argument. $\square$

Lemma 2.4.2. Let $\mathbf{x}^0$ belong to $\tilde{\mathcal{X}}(\Omega_0)$, $R > 0$, and $\mu > 0$. There exists a constant $C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) > 0$ such that, for all $0 < T < T_0$ and all $(\overline{\mathbf{u}}, p, \eta)$ in $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$, the following estimates hold

$$\|\overline{\mathbf{u}}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} + \|\overline{\mathbf{u}}\|_{L^4(0,T;\mathbf{L}^\infty(\Omega_0))} + \|\eta_t\|_{L^\infty(0,T;H^1(\Gamma_s))}$$
$$+ \|\eta_t\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{xx}\|_{L^\infty(0,T;H^1(\Gamma_s))} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}).$$

Proof. Let $(\mathring{\mathbf{u}}, \mathring{\eta}, \mathring{p})$ be the solution to (2.2.7) on the time interval $[0, T_0]$ with right-hand side 0 and the initial condition $(\mathbf{u}^0, \eta_1^0, \eta_2^0)$. We have

$$\|\overline{\mathbf{u}}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} \leq \left\|\overline{\mathbf{u}} - \mathring{\mathbf{u}}\right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} + \left\|\mathring{\mathbf{u}}\right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))},$$

then using that $\mathring{\mathbf{u}}(0) = \overline{\mathbf{u}}(0)$ and Lemma 2.4.1 the following estimate holds with the constant $C_0 = C_0(T_0)$

$$\begin{aligned}
\left\|\overline{\mathbf{u}} - \mathring{\mathbf{u}}\right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} &\leq C_0 \left\|\overline{\mathbf{u}} - \mathring{\mathbf{u}}\right\|_{\mathbf{H}^{2,1}(Q_T)} \\
&\leq C_0 \|\overline{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_T)} + C_0 \left\|\mathring{\mathbf{u}}\right\|_{\mathbf{H}^{2,1}(Q_T)} \\
&\leq C_0 R + C_0 \left\|\mathring{\mathbf{u}}\right\|_{\mathbf{H}^{2,1}(Q_{T_0})}.
\end{aligned}$$

The second part is estimated as follows

$$\left\|\mathring{\mathbf{u}}\right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} \leq \left\|\mathring{\mathbf{u}}\right\|_{L^\infty(0,T_0;\mathbf{H}^1(\Omega_0))} \leq C_0' \left\|\mathring{\mathbf{u}}\right\|_{\mathbf{H}^{2,1}(Q_{T_0})}.$$

Finally estimate (2.3.8) on $\mathring{\mathbf{u}}$ implies

$$\|\overline{\mathbf{u}}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} \leq C_0 R + (C_0 + C_0') C_L \left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)}.$$

The estimates on η_t and η_{xx} follow similarly. □

Lemma 2.4.3. Set $\mu_0 = \|(1+\eta^0)^{-1}\|_{L^\infty(\Gamma_s)}$. For $\eta \in H^{4,2}(\Sigma_T^s)$ such that $\eta(0) = \eta^0$ the function $\widetilde{\eta} = \frac{\eta - \eta^0}{1+\eta^0}$ satisfies the following estimates

$$\|\widetilde{\eta}\|_{L^\infty(\Sigma_T^s)} \leq \mu_0 T^{1/2} \|\eta_t\|_{L^2(0,T;L^\infty(\Gamma_s))}, \tag{2.4.1}$$

$$\|\widetilde{\eta}_x\|_{L^\infty(\Sigma_T^s)} \leq \mu_0 T^{1/2} \|\eta_{tx}\|_{L^2(0,T,L^\infty(\Gamma_s))} + \mu_0^2 \left\|\eta_x^0\right\|_{L^\infty(\Gamma_s)} T^{1/2} \|\eta_t\|_{L^2(0,T;L^\infty(\Gamma_s))}. \tag{2.4.2}$$

Proof. The estimates come from the fundamental theorem of calculus and Cauchy-Schwarz inequality. □

Lemma 2.4.4. Let u be in $L^\infty(0,T;H^1(\Omega_0))$ and v be in $H^{2,1}(Q_T)$. The following estimate holds

$$\|u \partial_i v\|_{L^2(Q_T)} \leq C T^{1/4} \|u\|_{L^\infty(0,T;H^1(\Omega_0))} \|v\|_{L^\infty(0,T;H^1(\Omega_0))}^{1/2} \|v\|_{L^2(0,T,H^2(\Omega_0))}^{1/2}, \tag{2.4.3}$$

with C independent of T, and $i = 1, 2$.

Proof. We have

$$\int_{\Omega_0} |u|^2 |\partial_1 v|^2 \leq \left(\int_{\Omega_0} |u|^6\right)^{1/3} \left(\int_{\Omega_0} |\partial_i v|^3\right)^{2/3},$$

and, using Lebesgue interpolation, $\|\partial_i v\|_{L^3(\Omega_0)} \leq \|\partial_i v\|_{L^2(\Omega_0)}^{1/2} \|\partial_i v\|_{L^6(\Omega_0)}^{1/2}$. Sobolev embeddings then yield

$$\|\partial_1 v\|_{L^4(0,T;L^3(\Omega_0))} \leq C \|v\|_{L^\infty(0,T;H^1(\Omega_0))}^{1/2} \|v\|_{L^2(0,T;H^2(\Omega_0))}^{1/2}.$$

Then we use the estimate $\|c\|_{L^2(0,T)} \leq T^{1/4} \|c\|_{L^4(0,T)}$ to obtain

$$\begin{aligned}
\|u\partial_i v\|_{L^2(0,T;L^2(\Omega_0))} &\leq \|u\|_{L^\infty(0,T;H^1(\Omega_0))} \|\partial_i v\|_{L^2(0,T;L^3(\Omega_0))} \\
&\leq T^{1/4} \|u\|_{L^\infty(0,T;H^1(\Omega_0))} \|\partial_i v\|_{L^4(0,T;L^3(\Omega_0))} \\
&\leq CT^{1/4} \|u\|_{L^\infty(0,T;H^1(\Omega_0))} \|v\|_{L^\infty(0,T;H^1(\Omega_0))}^{1/2} \|v\|_{L^2(0,T;H^2(\Omega_0))}^{1/2}.
\end{aligned}$$

□

Lemma 2.4.5. Let $\mathbf{x}^0$ belongs to $\tilde{\mathcal{X}}(\Omega_0)$, $R > 0$, and $\mu > 0$. For all $0 < T < T_0$ and $(\overline{\mathbf{u}}, p, \eta)$ in $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$, the function $M(\overline{\mathbf{u}}, \eta)$ belongs to $\mathbf{H}^{2,1}(Q_T)$ and the following estimate holds

$$\|M(\overline{\mathbf{u}}, \eta)\|_{\mathbf{H}^{2,1}(Q_T)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}). \tag{2.4.4}$$

Furthermore, for all $(\overline{\mathbf{u}}_1, p_1, \eta_1)$ and $(\overline{\mathbf{u}}_2, p_2, \eta_2)$ belonging to $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$, the following Lipschitz estimate holds

(2.4.5)

$$\|M(\overline{\mathbf{u}}_1, \eta_1) - M(\overline{\mathbf{u}}_2, \eta_2)\|_{\mathbf{H}^{2,1}(Q_T)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) \|(\overline{\mathbf{u}}_1, p_1, \eta_1) - (\overline{\mathbf{u}}_2, p_2, \eta_2)\|_{\mathcal{Y}_T}.$$

Proof. Through what follows we use the following basic estimate

$$\left\|(1 + \widetilde{\eta})^{-1}\right\|_{L^\infty(\Sigma_T^s)} = \left\|\frac{1 + \eta^0}{1 + \eta}\right\|_{L^\infty(\Sigma_T^s)} \leq \mu \left\|1 + \eta^0\right\|_{L^\infty(\Gamma_s)}.$$

Most of the estimates of $M(\overline{\mathbf{u}}, \eta) = \left(\dfrac{\overline{u}_1}{1 + \widetilde{\eta}}, \dfrac{z\widetilde{\eta}_x}{1 + \widetilde{\eta}}\overline{u}_1 + \overline{u}_2\right)^T$ and its derivatives are explicit $L^\infty \times L^2$ estimates using the previous lemmas and the regularity of $\widetilde{\eta}$. To estimate the second spatial derivative of $\frac{\overline{u}_1}{1+\widetilde{\eta}}$ we compute $\widetilde{\eta}_{xx}$:

$$\widetilde{\eta}_{xx} = \left(\frac{\eta_{xx} - \eta^0_{xx}}{1 + \eta^0}\right) - \left(\frac{2(\eta_x - \eta^0_x)\eta^0_x - (\eta - \eta^0)\eta^0_{xx}}{(1 + \eta^0)^2}\right) + \frac{2(\eta^0_x)^2(\eta - \eta^0)}{(1 + \eta^0)^4}.$$

Then

$$\begin{aligned}
\|\widetilde{\eta}_{xx}\|_{L^\infty(0,T;H^1(\Gamma_0))} &\leq C_1(\eta^0) + C_2(\eta^0) \|\eta_{xx}\|_{L^\infty(0,T;H^1(\Gamma_s))} \\
&\quad + C_3(\eta^0) \|\eta_x\|_{L^\infty(0,T;H^1(\Gamma_s))} + C_4(\eta^0) \|\eta\|_{L^\infty(0,T;H^1(\Gamma_s))} \\
&\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}).
\end{aligned}$$

This estimate is more precise that the one needed here (it implies an estimate on $\|\widetilde{\eta}_{xx}\|_{L^\infty(\Sigma_T^s)}$ using spatial Sobolev embeddings); we stated it because it is used in the estimates of Theorem 2.4.1. For the time derivative we have

$$\left\| \frac{-\widetilde{\eta}_t}{(1+\widetilde{\eta})^2} \right\|_{L^\infty(\Sigma_T^s)} \leq \mu^2 \left\| 1+\eta^0 \right\|^2_{L^\infty(\Sigma_T^s)} \mu \left\| \eta_t \right\|_{L^\infty(\Sigma_T^s)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}).$$

It follows from these estimates that

$$\left\| \frac{\overline{u}_1}{1+\widetilde{\eta}} \right\|_{H^{2,1}(Q_T)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}).$$

The second component of $M(\overline{\mathbf{u}}, \eta)$ and its derivatives estimated similarly, except for the terms

$$\left\| \frac{z\widetilde{\eta}_{xxx}\overline{u}_1}{1+\widetilde{\eta}} \right\|_{L^2(Q_T)} \leq C(\mu, \eta^0) \left\| \widetilde{\eta}_{xxx}\overline{u}_1 \right\|_{L^2(Q_T)}.$$

The term $\widetilde{\eta}_{xxx}$ is only in $L^2(\Gamma_s)$ and we cannot use Lemma 2.4.4. Let us write $\widetilde{\eta} = ND$ with $N = \eta - \eta^0$ and $D = (1+\eta^0)^{-1}$. We have

$$\widetilde{\eta}_{xxx} = N_{xxx}D + 3N_{xx}D_x + 3N_xD_{xx} + ND_{xxx}.$$

When multiplied by $\dfrac{z\overline{u}_1}{1+\widetilde{\eta}}$, the terms involving up to two derivatives can be estimated directly. For

$$\frac{z(\eta_{xxx} - \eta^0_{xxx})\overline{u}_1}{(1+\eta^0)(1+\widetilde{\eta})},$$

we have

$$\begin{aligned}
\left\| \frac{z(\eta_{xxx} - \eta^0_{xxx})\overline{u}_1}{(1+\eta^0)(1+\widetilde{\eta})} \right\|^2_{L^2(Q_T)} &\leq C(\mu, \eta^0) \int_0^T \left\| (\eta_{xxx} - \eta^0_{xxx})(\cdot, t) \right\|^2_{L^2(\Gamma_s)} \|\overline{\mathbf{u}}(\cdot, t)\|^2_{\mathbf{L}^\infty(\Omega_0)} \, dt \\
&\leq C(\mu, \eta^0) \left\| \eta_{xxx} - \eta^0_{xxx} \right\|^2_{L^4(0,T;L^2(\Gamma_s))} \|\overline{\mathbf{u}}\|^2_{L^4(0,T;\mathbf{L}^\infty(\Omega_0))} \\
&\leq C(\mu, \eta^0) T^{1/4} \left\| \eta_{xxx} - \eta^0_{xxx} \right\|^2_{L^8(0,T;L^2(\Gamma_s))} \|\overline{\mathbf{u}}\|^2_{L^4(0,T;\mathbf{L}^\infty(\Omega_0))},
\end{aligned}$$

and

$$\begin{aligned}
\left\| \eta_{xxx} - \eta^0_{xxx} \right\|_{L^8(0,T;L^2(\Gamma_s))} &\leq \|\eta_{xxx}\|_{L^8(0,T;L^2(\Gamma_s))} + T^{1/8} \left\| \eta^0 \right\|_{H^3(\Gamma_s)} \\
&\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) + T_0^{1/8} \left\| \eta^0 \right\|_{H^3(\Gamma_s)}.
\end{aligned}$$

This implies

$$\left\| \frac{z\widetilde{\eta}_{xxx}\overline{u}_1}{1+\widetilde{\eta}} \right\|_{L^2(Q_T)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}).$$

Thus (2.4.4) is proved. For the Lipschitz estimate we use the same techniques. Let us make explicit the estimate on one of the terms, namely

$$\frac{z\widetilde{\eta}_{1,x}}{1+\widetilde{\eta}_1}\overline{u}_{1,1} - \frac{z\widetilde{\eta}_{2,x}}{1+\widetilde{\eta}_2}\overline{u}_{2,1} = z\left(\frac{\widetilde{\eta}_{1,x}}{1+\widetilde{\eta}_1} - \frac{\widetilde{\eta}_{2,x}}{1+\widetilde{\eta}_2}\right)\overline{u}_{1,1} + \frac{z\widetilde{\eta}_{2,x}}{1+\widetilde{\eta}_2}(\overline{u}_{1,1} - \overline{u}_{2,1}).$$

Using the previous techniques and Lemma 2.4.1 we obtain

$$\left\| \frac{z\widetilde{\eta}_{2,x}}{1+\widetilde{\eta}_2}(\overline{u}_{1,1} - \overline{u}_{2,1}) \right\|_{H^{2,1}(Q_T)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) \|\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\|_{\mathbf{H}^{2,1}(Q_T)}.$$

For the other term we write

$$\frac{\widetilde{\eta}_{1,x}}{1+\widetilde{\eta}_1} - \frac{\widetilde{\eta}_{2,x}}{1+\widetilde{\eta}_2} = \frac{\widetilde{\eta}_{1,x} - \widetilde{\eta}_{2,x}}{(1+\widetilde{\eta}_1)(1+\widetilde{\eta}_2)} + \frac{\widetilde{\eta}_{1,x}(\widetilde{\eta}_2 - \widetilde{\eta}_1)}{(1+\widetilde{\eta}_1)(1+\widetilde{\eta}_2)} + \frac{\widetilde{\eta}_1(\widetilde{\eta}_{1,x} - \widetilde{\eta}_{2,x})}{(1+\widetilde{\eta}_1)(1+\widetilde{\eta}_2)},$$

and

$$\left\| z\left(\frac{\widetilde{\eta}_{1,x}}{1+\widetilde{\eta}_1} - \frac{\widetilde{\eta}_{2,x}}{1+\widetilde{\eta}_2}\right)\overline{u}_{1,1} \right\|_{H^{2,1}(Q_T)} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) \|\eta_1 - \eta_2\|_{H^{4,2}(\Sigma_T^s)}.$$

□

The nonlinearities in (2.2.3) can now be estimated.

Theorem 2.4.1. Let $\mathbf{x}^0$ belong to $\widetilde{\mathcal{X}}(\Omega_0)$, $R > 0$, and $\mu > 0$. There exists a function $P_{\theta,n}(T) = \sum_{k=0}^n T^{\theta_k}$ with $n \in \mathbb{N}^*$ and $\theta \in (\mathbb{R}_+^*)^{n+1}$ such that, for all $0 < T < T_0$ and all $(\overline{\mathbf{u}}, p, \eta) \in \mathcal{B}(\mathbf{x}^0, R, \mu, T)$, $(\mathbf{F}(\overline{\mathbf{u}}, p, \eta), \Theta(\overline{\mathbf{u}}), H(\overline{\mathbf{u}}, \eta))$ belongs to W_T and the following estimate holds

$$\|(\mathbf{F}(\overline{\mathbf{u}}, p, \eta), \Theta(\overline{\mathbf{u}}), H(\overline{\mathbf{u}}, \eta))\|_{W_T} \leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T). \tag{2.4.6}$$

Moreover, for $(\overline{\mathbf{u}}_i, p_i, \eta_i) \in \mathcal{B}(\mathbf{x}^0, R, \mu, T)$ $(i = 1, 2)$ the following estimate holds

$$\begin{aligned} &\|(\mathbf{F}_1, \Theta_1, H_1) - (\mathbf{F}_2, \Theta_2, H_2)\|_{W_T} \\ &\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T) \|(\overline{\mathbf{u}}_1, p_1, \eta_1) - (\overline{\mathbf{u}}_2, p_2, \eta_2)\|_{\mathcal{Y}_T}, \end{aligned} \tag{2.4.7}$$

with the notations $(\mathbf{F}_i, \Theta_i, H_i) = (\mathbf{F}(\overline{\mathbf{u}}_i, p_i, \eta_i), \Theta(\overline{\mathbf{u}}_i), H(\overline{\mathbf{u}}_i, \eta_i))$.

Proof. Step 1: Estimate of $\mathbf{F}(\overline{\mathbf{u}}, p, \eta)$. We recall the form of $\mathbf{F}(\overline{\mathbf{u}}, p, \eta)$:

$$\mathbf{F}(\overline{\mathbf{u}}, p, \eta) = \mathbf{G}(M(\overline{\mathbf{u}}, \eta), p, \eta) - \partial_t N(\overline{\mathbf{u}}, \eta) + \nu \Delta N(\overline{\mathbf{u}}, \eta).$$

Set $\mathbf{u} = M(\overline{\mathbf{u}}, \eta)$. Thanks to Lemma 2.4.5 we can prove the estimates with $\mathbf{u}$ and then obtain estimates in terms of $\overline{\mathbf{u}}$. For

$$\begin{aligned} \mathbf{G}(\mathbf{u}, p, \eta) = {} & -\widetilde{\eta}\mathbf{u}_t + \left[z\widetilde{\eta}_t + \nu z \left(\frac{\widetilde{\eta}_x^2}{1+\widetilde{\eta}} - \widetilde{\eta}_{xx} \right) \right] \mathbf{u}_z \\ & + \nu \left[-2z\widetilde{\eta}_x \mathbf{u}_{xz} + \widetilde{\eta}\mathbf{u}_{xx} + \frac{z^2\widetilde{\eta}_x^2 - \widetilde{\eta}}{1+\widetilde{\eta}} \mathbf{u}_{zz} \right] \\ & + z(\widetilde{\eta}_x p_z - \widetilde{\eta} p_x)\mathbf{e}_1 - (1+\widetilde{\eta})u_1 \mathbf{u}_x + (z\widetilde{\eta}_x u_1 - u_2)\mathbf{u}_z, \end{aligned}$$

we use L^∞ estimates on $\widetilde{\eta}$ to gain a factor T for terms like the first one:

$$\begin{aligned}\|-\widetilde{\eta}\mathbf{u}_t\|_{\mathbf{L}^2(Q_T)} &\le \|\widetilde{\eta}\|_{L^\infty(\Sigma_T^s)}\|\mathbf{u}_t\|_{\mathbf{L}^2(Q_T)}\\ &\le \mu T^{1/2}\|\eta_t\|_{L^2(0,T;L^\infty(\Gamma_s))}\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)} \le C(T_0,R,\mu,\|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)})T^{1/2}.\end{aligned}$$

For the product of functions in $L^\infty(0,T;H^1(\Omega_0))$ with derivatives of functions in $H^{2,1}(Q_T)$ we use Lemma 2.4.4, for example:

$$\begin{aligned}\|z\widetilde{\eta}_t\mathbf{u}_z\|_{\mathbf{L}^2(Q_T)} &\le CT^{1/4}\|\widetilde{\eta}_t\|_{L^\infty(0,T;H^1(\Gamma_s))}\|\mathbf{u}\|^{1/2}_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))}\|\mathbf{u}\|^{1/2}_{L^2(0,T;\mathbf{H}^2(\Omega_0))}\\ &\le C(T_0,R,\mu,\|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)})T^{1/4},\end{aligned}$$

and

$$\begin{aligned}\|-\nu z\widetilde{\eta}_{xx}\mathbf{u}_z\|_{\mathbf{L}^2(Q_T)} &\le CT^{1/4}\|\widetilde{\eta}_{xx}\|_{L^\infty(0,T;H^1(\Omega_0))}\|\mathbf{u}\|^{1/2}_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))}\|\mathbf{u}\|^{1/2}_{L^2(0,T;\mathbf{H}^2(\Omega_0))}\\ &\le C(T_0,R,\mu,\|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)})T^{1/4},\end{aligned}$$

where we have used the estimates on $\widetilde{\eta}_t$ and $\widetilde{\eta}_{xx}$ given in the proof of Lemma 2.4.5.

The term $N(\overline{\mathbf{u}},\eta) = \left(\dfrac{-\widetilde{\eta}\overline{u}_1}{1+\widetilde{\eta}}, \dfrac{z\widetilde{\eta}_x\overline{u}_1}{1+\widetilde{\eta}}\right)^T$ has already been estimated in the proof of Lemma 2.4.5 and the factor T is obtained with the previous techniques and Lebesgue interpolation for the terms

$$\frac{z(\eta_{xxx}-\eta^0_{xxx})\overline{u}_1}{(1+\eta^0)(1+\widetilde{\eta})}.$$

Step 2: Estimate of $\Theta(\overline{\mathbf{u}})$. In order to obtain an estimate in $L^2(0,T;H^{1/2}(\Gamma_{i,o}))$ we study $\Theta(\overline{\mathbf{u}}) = (1/2)|\overline{\mathbf{u}}|^2$ on Ω_0 and then look for the restriction to $\Gamma_{i,o}$. We have

$$\begin{aligned}\Theta(\overline{\mathbf{u}}) &= \frac{\overline{u}_1^2+\overline{u}_2^2}{2},\\ \Theta(\overline{\mathbf{u}})_x &= \overline{u}_1\overline{u}_{1,x}+\overline{u}_2\overline{u}_{2,x},\\ \Theta(\overline{\mathbf{u}})_z &= \overline{u}_1\overline{u}_{1,z}+\overline{u}_2\overline{u}_{2,z},\end{aligned}$$

and using Lemma 2.4.4

$$\|\overline{u}_1\overline{u}_{1,x}\|_{L^2(Q_T)} \le CT^{1/4}\|\overline{\mathbf{u}}\|^{3/2}_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))}\|\overline{\mathbf{u}}\|_{L^2(0,T;\mathbf{H}^2(\Omega_0))} \le C(T_0,R,\mu,\|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)})T^{1/4}.$$

which implies a $L^2(0,T;H^1(\Omega_0))$ estimate on $\Theta(\overline{\mathbf{u}})$ and thus a $L^2(0,T;H^{1/2}(\Gamma_{i,o}))$ estimate for the trace.

Step 3: Estimate of $H(\overline{\mathbf{u}},\eta)$. We recall that $H(\overline{\mathbf{u}},\eta) = \Psi(M(\overline{\mathbf{u}},\eta),\eta)$ with

$$\Psi(\mathbf{u},\eta) = \nu\left(\frac{\eta_x}{1+\widetilde{\eta}}u_{1,z} + \eta_x u_{2,x} - \frac{\widetilde{\eta}_x\eta_x z-2}{1+\widetilde{\eta}}u_{2,z}\right).$$

For the terms without $\widetilde{\eta}$ we use directly the regularity of $\mathbf{u}$ to gain a factor T. Using fractional Sobolev embeddings [1, Theorem 7.58] and trace theorems we know that $\mathbf{u}_x(x, 1+\eta^0(x), t), \mathbf{u}_z(x, 1+\eta^0(x), t)$ belong to $H^{1/2,1/4}(Q_T)$ and $L^4(0,T;\mathbf{L}^2(\Gamma_s))$. Hence

$$\left\| \nu \frac{\eta_x}{1+\widetilde{\eta}} u_{1,z} \right\|_{L^2(0,T;L^2(\Gamma_s))} \leq C(\mu, \eta^0) \left\| \eta_x u_{1,z} \right\|_{L^2(0,T;L^2(\Gamma_s))},$$

and

$$\begin{aligned}
\|\eta_x u_{1,z}\|^2_{L^2(\Sigma^s_T)} &\leq \|\eta_x\|^2_{L^\infty(\Sigma^s_T)} \int_0^T \|u_{1,z}\|^2_{L^2(\Gamma_s)}\, dt \\
&\leq \|\eta_x\|^2_{L^\infty(\Sigma^s_T)} T^{1/2} \|u_{1,z}\|^2_{L^4(0,T;L^2(\Gamma_s))} \\
&\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) T^{1/2}.
\end{aligned}$$

The other terms are estimated with the previous techniques.

Step 4: Lipschitz estimates. The Lipschitz estimates are obtained with the same techniques. Let us make explicit some inequalities.

$$\begin{aligned}
&\|\widetilde{\eta}_1 \mathbf{u}_{1,t} - \widetilde{\eta}_2 \mathbf{u}_{2,t}\|_{\mathbf{L}^2(Q_T)} \\
&\leq \|(\widetilde{\eta}_1 - \widetilde{\eta}_2)\mathbf{u}_{1,t}\|_{\mathbf{L}^2(Q_T)} + \|\widetilde{\eta}_2(\mathbf{u}_{1,t} - \mathbf{u}_{2,t})\|_{\mathbf{L}^2(Q_T)} \\
&\leq \|\widetilde{\eta}_1 - \widetilde{\eta}_2\|_{L^\infty(\Sigma^s_T)} \|\mathbf{u}_{1,t}\|_{\mathbf{L}^2(Q_T)} + \|\widetilde{\eta}_2\|_{L^\infty(\Sigma^s_T)} \|\mathbf{u}_{1,t} - \mathbf{u}_{2,t}\|_{\mathbf{L}^2(Q_T)} \\
&\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) T^{1/2} (\|\eta_1 - \eta_2\|_{H^{4,2}(\Sigma^s_T)} + \|\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\|_{\mathbf{H}^{2,1}(Q_T)}).
\end{aligned}$$

All the interest of working in the initial domain Ω_0 instead of the rectangular Ω comes from the estimate (2.4.3) on $\widetilde{\eta}$. With the usual change of variables, the term $\eta_2(\mathbf{u}_1 - \mathbf{u}_2)$ cannot be estimated without smallness assumption on η^0. For $\nu z \widetilde{\eta}_{1,xx}\mathbf{u}_{1,z} - \nu z \widetilde{\eta}_{2,xx}\mathbf{u}_{2,z}$ we have

$$\begin{aligned}
&\|\nu z(\widetilde{\eta}_{1,xx} - \widetilde{\eta}_{2,xx})\mathbf{u}_{1,z}\|_{\mathbf{L}^2(Q_T)} \\
&\leq C T^{1/4} \|\widetilde{\eta}_{1,xx} - \widetilde{\eta}_{2,xx}\|_{L^\infty(0,T;H^1(\Omega_0))} \|\mathbf{u}_1\|^{1/2}_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} \|\mathbf{u}_1\|^{1/2}_{L^2(0,T;\mathbf{H}^2(\Omega_0))} \\
&\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) T^{1/4} \|\eta_1 - \eta_2\|_{H^{4,2}(\Sigma^s_T)},
\end{aligned}$$

and

$$\begin{aligned}
&\|\nu z \widetilde{\eta}_{2,xx}(\mathbf{u}_{1,z} - \mathbf{u}_{2,z})\|_{\mathbf{L}^2(Q_T)} \\
&\leq C T^{1/4} \|\widetilde{\eta}_{2,xx}\|_{L^\infty(0,T;H^1(\Omega_0))} \|\mathbf{u}_1 - \mathbf{u}_2\|^{1/2}_{L^\infty(0,T;\mathbf{H}^1(\Omega_0))} \|\mathbf{u}_1 - \mathbf{u}_2\|^{1/2}_{L^2(0,T;\mathbf{H}^2(\Omega_0))} \\
&\leq C(T_0, R, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) T^{1/4} \|\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\|_{\mathbf{H}^{2,1}(Q_T)}.
\end{aligned}$$

The Lipschitz estimates with lower regularity terms like $\widetilde{\eta}_{xxx}$ are obtained as in the proof of Lemma 2.4.5. $\square$

2.4.2 Fixed point procedure

For all $\mathbf{x}^0 = (\mathbf{u}^0, \eta_1^0, \eta_2^0) \in \tilde{\mathcal{X}}(\Omega_0)$, $R > 0$, $\mu > 0$ and $T > 0$ such that $\mathcal{B}(\mathbf{x}^0, R, \mu, T) \neq \emptyset$, consider the map

$$\mathcal{F} : \left\{ \begin{array}{l} \mathcal{B}(\mathbf{x}^0, R, \mu, T) \longrightarrow \mathcal{Y}_T, \\ (\overline{\mathbf{u}}, p, \eta) \mapsto (\overline{\mathbf{u}}^*, p^*, \eta^*), \end{array} \right. \tag{2.4.8}$$

where $(\overline{\mathbf{u}}^*, p^*, \eta^*)$ is the solution to (2.2.7) with right-hand side $(\mathbf{F}(\overline{\mathbf{u}}, p, \eta), \Theta(\overline{\mathbf{u}}), H(\overline{\mathbf{u}}, \eta))$. In order to solve (2.2.6) we look for a fixed point of the map $\mathcal{F}$.

Theorem 2.4.2. For all $\mathbf{x}^0 = (\mathbf{u}^0, \eta_1^0, \eta_2^0)$ in $\tilde{\mathcal{X}}(\Omega_0)$, there exist $R > 0$, $\mu > 0$ and $T > 0$ such that $\mathcal{F}$ is a contraction from $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$ into $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$. Hence $\mathcal{F}$ has a fixed point.

Proof. Set $\mu = \|(1 + \eta^0)^{-1}\|_{L^\infty(\Gamma_s)}$. Let $(\mathring{\mathbf{u}}, \mathring{\eta}, \mathring{p})$ be the solution on $[0, T_0]$ to (2.2.7) with right-hand side 0. We choose R_1 such that $\left\|(\mathring{\mathbf{u}}, \mathring{\eta}, \mathring{p})\right\|_{\mathcal{X}_{T_0}} \leq R_1$. Writing

$$1 + \mathring{\eta}(t) = 1 + \eta^0 + \int_0^t \mathring{\eta}_t(s) ds,$$

we can choose $0 < T_1 < T_0$ such that

$$\left\|(1 + \mathring{\eta})^{-1}\right\|_{L^\infty(\Sigma_{T_1}^s)} \leq \frac{1}{\|1 + \eta^0\|_{L^\infty(\Gamma_s)} - T_1 C(\mu, T_0, R_1, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)})} \leq 2\mu. \tag{2.4.9}$$

Thus the ball $\mathcal{B}(\mathbf{x}^0, R_1, \mu, T_1)$ is non-empty. From Theorems 2.4.1 and 2.3.4, it follows that

$$\|\mathcal{F}(\overline{\mathbf{u}}, p, \eta)\|_{\mathcal{Y}_{T_1}} \leq C_L \left(\left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)} + C(T_0, R_1, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T_1) \right),$$

for all $(\overline{\mathbf{u}}, p, \eta) \in \mathcal{B}(\mathbf{x}^0, R_1, \mu, T_1)$.

Then we choose $R_2 \geq R_1$ such that $R_2 \geq 2C_L \|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\|_{\mathcal{X}(\Omega_0)}$, and $T_2 \leq T_1$ such that

$$C_L C(T_0, R_2, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T_2) \leq C_L \left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)} \text{ and } \left\|(1 + \eta^*)^{-1}\right\|_{L^\infty(\Sigma_{T_2}^s)} \leq 2\mu.$$

Therefore $\mathcal{F}$ is well defined from $\mathcal{B}(\mathbf{x}^0, R_2, \mu, T_2)$ into $\mathcal{B}(\mathbf{x}^0, R_2, \mu, T_2)$. Still with Theorems 2.4.1 and 2.3.4, it follows that

$$\begin{aligned} &\|\mathcal{F}(\overline{\mathbf{u}}_1, p_1, \eta_1) - \mathcal{F}(\overline{\mathbf{u}}_2, p_2, \eta_2)\|_{\mathcal{Y}_{T_2}} \\ &\leq C_L C(T_0, R_2, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T_2) \|(\overline{\mathbf{u}}_1, p_1, \eta_1) - (\overline{\mathbf{u}}_2, p_2, \eta_2)\|_{\mathcal{Y}_{T_2}}, \end{aligned}$$

for all $(\overline{\mathbf{u}}_1, p_1, \eta_1)$ and $(\overline{\mathbf{u}}_2, p_2, \eta_2)$ belonging to $\mathcal{B}(\mathbf{x}^0, R_2, \mu, T_2)$. We choose $0 < T_3 \leq T_2$ such that $C_L C(T_0, R_2, \mu, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T_3) \leq 1/2$. The mapping $\mathcal{F}$ is a contraction from the complete metric space $\mathcal{B}(\mathbf{x}^0, R_2, \mu, T_3)$ into itself, and the Banach fixed point theorem concludes the proof. □

Theorem 2.4.3. For all $(\mathbf{u}^0, \eta_1^0, \eta_2^0) \in \tilde{\mathcal{X}}(\Omega_0)$ there exists $T > 0$ such that the system (2.2.6) has a unique strong solution $(\overline{\mathbf{u}}, p, \eta)$ in $\mathbf{H}^{2,1}(Q_T) \times L^2(0,T; H^1(\Omega_0)) \times H^{4,2}(\Sigma_T^s)$.

Proof. The existence is already proved. Set $\mathbf{x}^0 = (\mathbf{u}^0, \eta_1^0, \eta_2^0)$ and let $(\overline{\mathbf{u}}, p, \eta)$ be the unique solution to (2.2.6) in $\mathcal{B}(\mathbf{x}^0, R, \mu, T)$ with

$$\mu = \left\|(1+\eta^0)^{-1}\right\|_{L^\infty(\Gamma_s)}, \quad R = 2C_L \left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)},$$

and $T > 0$, constructed by the fix point method in the previous Theorem. Let $(\overline{\mathbf{u}}', p', \eta')$ be another solution to (2.2.6) defined on $[0,T]$ with the same initial data. Define the constants $R_0 = \|(\overline{\mathbf{u}}', p', \eta')\|_{\mathcal{Y}_T}$ and $\mu_0 = \|(1+\eta')^{-1}\|_{L^\infty(\Sigma_T^s)}$. Assume that $T > 0$ is small enough such that $\|(1+\eta')^{-1}\|_{L^\infty(\Sigma_T^s)} \leq 2\mu$. From Theorems 2.4.1 and 2.3.4, it follows that

$$\|(\overline{\mathbf{u}}', p', \eta')\|_{\mathcal{Y}_{T_1}} \leq C_L \left(\left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)} + C(T_0, R_0, \mu_0, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T_1) \right),$$

for all $0 < T_1 \leq T$. Let us choose $0 < T_1 \leq T$ such that

$$C_L C(T_0, R_0, \mu_0, \|\mathbf{x}^0\|_{\mathcal{X}(\Omega_0)}) P_{\theta,n}(T_1) \leq C_L \left\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\right\|_{\mathcal{X}(\Omega_0)}.$$

Hence $(\overline{\mathbf{u}}', p', \eta')$ belongs to $\mathcal{B}(\mathbf{x}^0, R, \mu, T_1)$ and $(\overline{\mathbf{u}}, p, \eta) = (\overline{\mathbf{u}}', p', \eta')$ on $[0, T_1]$.

Let $0 < T^* \leq T$ be the greatest time such that the two solutions are equal. We then consider the system (2.2.6) starting at the time T^*, rewritten in $\Omega_{\eta(T^*)}$, with the initial conditions $(\overline{\mathbf{u}}(T^*), \eta(T^*), \eta_t(T^*)) = (\overline{\mathbf{u}}'(T^*), \eta'(T^*), \eta_t'(T^*))$. If $T^* < T$, using the fixed point procedure we prove the existence of a solution $(\overline{\mathbf{u}}'', \eta'', \eta_t'')$ on $[T^*, T_2]$ with $T_2 > T^*$. The previous argument shows that there exists $T_3 > 0$ such that the three solutions are equal (after a change of variable in order to consider functions in the domain $\Omega_{\eta(T^*)}$) on $[T^*, T_3]$ which is a contradiction with the definition of T^*. Hence $T^* = T$ and the solution to (2.2.6) is unique. $\square$

The previous ideas and techniques can be applied on system (2.1.1) with the Dirichlet boundary conditions $\mathbf{u} = 0$ on $\Gamma_{i,o}$ and thus fix the gap in the proof of local existence in [37].

To conclude this section, we state the existence and uniqueness of a solution for (2.2.6) on $[0,T]$, with $T > 0$ a fixed time and smallness assumptions on the initial data. This result is proved on the rectangular domain since the estimates of the nonlinear terms are done through the radius of the ball in the fixed point argument. The existence technique is similar to the one in [56, Theorem 10.1] and the uniqueness comes from the local existence and uniqueness result. Let us notice that with this approach the nonlinear term in the beam equation in (2.2.3) with $\eta^0 = 0$ writes

$$\begin{aligned}
\Psi(\widehat{\mathbf{u}}, \eta) &= \nu \left(\frac{\eta_x}{1+\eta} \widehat{u}_{1,z} + \eta_x \widehat{u}_{2,x} - \frac{\eta_x^2 z - 2}{1+\eta} \widehat{u}_{2,z} \right) \\
&= -2\nu \widehat{u}_{2,z} + \nu \left(\frac{\eta_x}{1+\eta} \widehat{u}_{1,z} + \eta_x u_{2,x} - \frac{\eta_x^2 z - 2\eta}{1+\eta} \widehat{u}_{2,z} \right) \\
&= -2\nu \widehat{u}_{2,z} + \overline{\Psi}(\widehat{\mathbf{u}}, \eta).
\end{aligned}$$

After writing $\widehat{\mathbf{u}} = M(\overline{\mathbf{u}}, \eta) = \overline{\mathbf{u}} + N(\overline{\mathbf{u}}, \eta)$ this nonlinear term becomes

$$H(\overline{\mathbf{u}}, \eta) = -2\nu \overline{u}_{2,z} - 2\nu N(\overline{\mathbf{u}}, \eta)_{2,z} + \overline{\Psi}(M(\overline{\mathbf{u}}, \eta), \eta),$$

and as div $\overline{\mathbf{u}} = \overline{u}_{1,x} + \overline{u}_{2,z} = 0$ in Q_T and $\overline{u}_{1,x} = 0$ on $\Sigma_T^s = \Sigma_T^0$ we obtain $\overline{u}_{2,z} = 0$ on Σ_T^s. Hence all the nonlinear terms in the beam equation are at least quadratic.

Theorem 2.4.4. Let $T > 0$ be a fixed time and recall that $\Omega = (0, L) \times (0, 1)$. There exists $r > 0$ such that for all $(\mathbf{u}^0, \eta_1^0, \eta_2^0)$ in $\mathcal{X}(\Omega)$ satisfying $\|(\mathbf{u}^0, \eta_1^0, \eta_2^0)\|_{\mathcal{X}(\Omega)} \leq r$, the system (2.2.6) admits a unique solution in $\mathbf{H}^{2,1}(Q_T) \times L^2(0, T; H^1(\Omega)) \times H^{4,2}(\Sigma_T^s)$.

Remark 2.4.1. Note that the initial condition is taken in $\mathcal{X}(\Omega)$, not $\tilde{\mathcal{X}}(\Omega)$, which means that η_0^1 can be different from 0.

2.5 Appendix

2.5.1 Steady Stokes equations

Consider the steady Stokes equations

$$\begin{aligned}
&-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \ \text{div}\ \mathbf{u} = 0 \ \text{ in } \Omega_0, \\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_0, \ \mathbf{u} = 0 \ \text{ on } \Gamma_b, \ u_2 = 0 \ \text{ and } \ p = h \ \text{ on } \Gamma_{i,o},
\end{aligned} \tag{2.5.1}$$

with $\mathbf{f} \in \mathbf{L}^2(\Omega_0)$, $\mathbf{g} = (0, g)^T \in \mathcal{H}_{00}^{3/2}(\Gamma_0)$ and $h \in H^{1/2}(\Gamma_{i,o})$. We prove in Theorem 2.5.4 the existence and uniqueness of a pair $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ solution to (2.5.1). An existence and uniqueness result for (2.5.1) with weaker data is given in Theorem 2.5.7. The nonhomogeneous boundary condition on the pressure is handled directly with a lifting operator $\mathcal{R} \in \mathcal{L}(H^{1/2}(\Gamma_{i,o}), H^1(\Omega_0))$. For the nonhomogeneous Dirichlet boundary condition we use the following theorem.

Theorem 2.5.1. For all $\mathbf{g} = (0, g)^T \in \mathcal{H}_{00}^{3/2}(\Gamma_0)$ the system

$$\begin{aligned}
&\text{div}\ \mathbf{w} = 0 \ \text{ in } \Omega_0, \\
&\mathbf{w} = \mathbf{g} \ \text{ on } \Gamma_0, \ \mathbf{w} = 0 \ \text{ on } \Gamma_b, \ w_2 = 0 \ \text{ on } \Gamma_{i,o},
\end{aligned} \tag{2.5.2}$$

admits a solution $\mathbf{w} \in \mathbf{H}^2(\Omega_0)$ satisfying the estimate

$$\|\mathbf{w}\|_{\mathbf{H}^2(\Omega_0)} \leq C \|\mathbf{g}\|_{\mathcal{H}_{00}^{3/2}(\Gamma_0)}.$$

Proof. We look for $\mathbf{w}$ under the form $\mathbf{w} = (-\partial_2\phi, \partial_1\phi)^T$, which ensures the property div $\mathbf{w} = 0$. The boundary conditions on $\mathbf{w}$ imply the following conditions on ϕ

$$\begin{array}{l} \partial_2\phi = 0 \text{ and } \partial_1\phi = g \text{ on } \Gamma_0, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \partial_1\phi = 0 \text{ on } \Gamma_{i,o},\ \partial_2\phi = \partial_1\phi = 0 \text{ on } \Gamma_b. \end{array} \tag{2.5.3}$$

Let η_e^0 be an $H^3(\mathbb{R})$ extension of η^0. We consider the change of variables

$$\psi^\pm : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x,y) \mapsto (x, y \pm \eta_e^0(x)). \end{cases}$$

Let $\widehat{v}$ be a function in $H^3(\mathbb{R}^2)$. Thanks to the H^3-regularity of η_e^0, the function $\widehat{v} \circ \psi^\pm$ is still in $H^3(\mathbb{R}^2)$. We search for ϕ solution to (2.5.3) under the form $\phi = \widehat{\phi} \circ \psi^-$ with $\widehat{\phi} \in H^3((0,L) \times (-\infty, 1))$ satisfying

$$\begin{array}{l} \partial_2\widehat{\phi} = 0 \text{ and } \partial_1\widehat{\phi} = \widehat{g} \text{ on } \Gamma_s, \\ \partial_1\widehat{\phi} = 0 \text{ on } \Gamma_{i,o},\ \widehat{\phi} = 0 \text{ on } (0,L) \times (-\infty, 1-\delta), \end{array} \tag{2.5.4}$$

with $\widehat{g} = g \circ \psi^+$ and

$$\delta = \begin{cases} \min\limits_{x\in(0,L)} (1 + \eta^0(x)) \text{ if } \min\limits_{x\in(0,L)} (1+\eta^0(x)) < 1, \\ \alpha \text{ if } \min\limits_{x\in(0,L)} (1+\eta^0(x)) \geq 1, \end{cases} \tag{2.5.5}$$

for a fixed $\alpha \in (0,1)$. This condition is used to ensure that the function $\phi = \widehat{\phi} \circ \psi^-$ is equal to zero near Γ_b, in order to fulfil the boundary conditions $\partial_1\phi = \partial_2\phi = 0$ on Γ_b. To build $\widehat{\phi}$ we first search for $\widehat{\phi}_o$ such that

$$\begin{array}{l} \frac{\partial\widehat{\phi}_o}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_s \cup \Gamma_o, \\ \widehat{\phi}_o(x,y) = G(x,y) = \int_0^x \widehat{g}(s)ds \text{ for } (x,y) \in \Gamma_s, \\ \widehat{\phi}_o = 0 \text{ on } (0,L) \times (-\infty, 1-\delta), \end{array} \tag{2.5.6}$$

The boundary conditions on Γ_s are handled directly thanks to a lifting and a symmetry argument is used to obtain the homogeneous Neumann boundary condition on Γ_o. We set

$$G^* : \begin{cases} G^*(x,y) = G(x,y) \text{ for } (x,y) \in \Gamma_s, \\ G^*(x,y) = G(2L - x, y) \text{ for } (x,y) \in (L, 2L) \times \{1\}. \end{cases}$$

Denote by $\widehat{g}_s$ the odd extension of $\widehat{g}$ on $\Gamma_{s,s} = (0,2L)\times\{1\}$. As $\widehat{g} \in H^{3/2}_{00}(\Gamma_s)$, the function $\widehat{g}_s$ belongs to $H^{3/2}(\Gamma_{s,s})$. Indeed odd and even symmetries preserve the H^1-regularity (resp. H^2-regularity) for functions in $H^1_0(\Gamma_0)$ (resp. in $H^2_0(\Gamma_0)$), thus, by interpolation, the $H^{3/2}$-regularity is also preserved for functions in $H^{3/2}_{00}(\Gamma_0) = [H^1_0(\Gamma_0), H^2_0(\Gamma_0)]_{1/2}$.

As $\partial_1 G^*(\cdot, 1) = \widehat{g}_s(\cdot)$ we have $G^* \in H^{5/2}(\Gamma_{s,s})$. We still denote by G^* a regular extension of G^* on $\mathbb{R} \times \{1\}$. The lifting results in [39] in the case of the half-plan give a function

$\widehat{\phi}_1 \in H^3(\mathbb{R} \times (-\infty, 1))$ such that $\widehat{\phi}_1 = G^*$ and $\frac{\partial \widehat{\phi}_1}{\partial \mathbf{n}} = 0$ on $\mathbb{R} \times \{1\}$. We then use cut-off functions to ensure that $\widehat{\phi}_1 = 0$ on $(0, 2L) \times (-\infty, 1-\delta)$.

Introduce the symmetric function $\widehat{\phi}_2$ to $\widehat{\phi}$ with respect to the axis $x = L$ defined by $\widehat{\phi}_2(x, y) = \widehat{\phi}_2(2L - x, y)$ for $(x, y) \in (0, 2L) \times (-\infty, 1)$. As the Dirichlet boundary condition G^* is symmetric, $\widehat{\phi}_2$ satisfies the same boundary conditions as $\widehat{\phi}_1$ on $\Gamma_{s,s}$. We finally set $\widehat{\phi}_o = \frac{\widehat{\phi}_1 + \widehat{\phi}_{1,s}}{2}$. The function $\widehat{\phi}_o$ belongs to $H^3((0, 2L) \times (-\infty, 1))$ and admits $x = L$ as an axis of symmetry. Hence we have $\frac{\partial \widehat{\phi}_o}{\partial \mathbf{n}} = 0$ on Γ_o and the restriction on $(0, L) \times (-\infty, 1)$ is a solution to (2.5.6).

Using the same tools we obtain a function $\widehat{\phi}_i \in H^3((0, L) \times (-\infty, 1))$ such that

$$\begin{aligned} &\tfrac{\partial \widehat{\phi}_i}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_s \cup \Gamma_i, \\ &\widehat{\phi}_i(x, y) = G(x, y) = \textstyle\int_0^x \widehat{g}(s) ds \text{ for } (x, y) \in \Gamma_s, \\ &\widehat{\phi}_i = 0 \text{ on } (0, L) \times (-\infty, 1-\delta). \end{aligned} \tag{2.5.7}$$

Then we combine $\widehat{\phi}_o$ and $\widehat{\phi}_i$. Let α be a function defined on $[0, L]$ such that $\alpha = 1$ near Γ_i, $\alpha = 0$ near Γ_o and $\alpha \in \mathcal{C}^\infty([0, L])$. The function $\widehat{\phi}$ defined by

$$\widehat{\phi}(x, y) = \alpha(x)\widehat{\phi}_i(x, y) + (1 - \alpha(x))\widehat{\phi}_o(x, y) \text{ for all } (x, y) \in (0, L) \times (-\infty, 1),$$

is a solution to (2.5.4). Finally the restriction to Ω_0 of the function $\phi = \widehat{\phi} \circ \psi^-$ is a solution to (2.5.3). Indeed,

$$\begin{aligned} &\partial_2 \phi = \partial_2 \widehat{\phi} \circ \psi^- = 0 \ \text{ on } \Gamma_0, \\ &\partial_1 \phi = \partial_1 \widehat{\phi} \circ \psi^- - \eta_x^0 \partial_2 \widehat{\phi} \circ \psi^- = \partial_1 \widehat{\phi} \circ \psi^- = \widehat{g} \circ \psi^- = g \text{ on } \Gamma_0, \\ &\tfrac{\partial \phi}{\partial \mathbf{n}} = \partial_1 \phi = 0 \text{ on } \Gamma_{i,o}, \ \partial_2 \phi = \partial_1 \phi = 0 \text{ on } \Gamma_b. \end{aligned}$$

and $\mathbf{w} = (-\partial_2 \phi, \partial_1 \phi)^T$ is a solution of (2.5.2). We have $\mathbf{w} \in \mathbf{H}^2(\Omega_0)$ and the estimate follows from the continuity of the lifting operator in [39]. $\square$

Let $\mathbf{w} \in \mathbf{H}^2(\Omega_0)$ be the lifting of $\mathbf{g}$ given by Theorem 2.5.1 and $H = \mathcal{R}(h)$. By setting $(\mathbf{v}, q) = (\mathbf{u}, p) - (\mathbf{w}, H)$ the Stokes system (2.5.1) is equivalent to

$$\begin{aligned} -\nu \Delta \mathbf{v} + \nabla q = \bar{\mathbf{f}}, \ \ \text{div } \mathbf{v} = 0 \ \text{ in } \Omega_0, \\ \mathbf{v} = 0 \ \text{ on } \Gamma_d, \ v_2 = 0 \ \text{ and } q = 0 \ \text{ on } \Gamma_{i,o}, \end{aligned} \tag{2.5.8}$$

with $\bar{\mathbf{f}} = \mathbf{f} + \nu \Delta \mathbf{w} - \nabla H$. Using Green formula one can derive the following variational formulation for (2.5.8).

Theorem 2.5.2. Let $(\mathbf{v}, q) \in \mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ be a solution to (2.5.8). Then $\mathbf{v}$ satisfies the variational formulation :

Find $\mathbf{v} \in V$ such that $\quad \nu \displaystyle\int_{\Omega_0} \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} = \int_{\Omega_0} \bar{\mathbf{f}} \cdot \boldsymbol{\varphi}$ for all $\boldsymbol{\varphi} \in V$. $\quad (\star)$

Theorem 2.5.3. The variational formulation ($\star$) admits a unique solution $\mathbf{v} \in V$. Moreover there exists a pressure $\mathcal{Q} \in L^2(\Omega_0)$, unique up to an additive constant, such that $-\nu\Delta\mathbf{v} + \nabla\mathcal{Q} = \bar{\mathbf{f}}$ in $\mathbf{H}^{-1}$.

The pressure $\mathcal{Q}$ is mentioned as a pressure associated with $\mathbf{v}$.

Proof. As the only constant in V is the null function we can use a Poincaré inequality to prove that the bilinear form

$$a(\mathbf{v}, \boldsymbol{\varphi}) = \nu \int_{\Omega_0} \nabla\mathbf{v} : \nabla\boldsymbol{\varphi},$$

is coercive on V. Hence the Lax-Milgram lemma gives us the existence of a unique solution $\mathbf{v} \in V$ to the variational formulation ($\star$). For the pressure, we use the equality

$$\left\langle -\nu\Delta\mathbf{v} - \bar{\mathbf{f}}, \boldsymbol{\varphi} \right\rangle_{\mathbf{H}^{-1},\mathbf{H}_0^1} = 0, \text{ for all } \boldsymbol{\varphi} \in (H_0^1(\Omega_0))^2 \text{ such that div } \boldsymbol{\varphi} = 0,$$

and [14, Chap 4, Theorem 2.3] to prove the existence of $\mathcal{Q} \in L^2(\Omega_0)$, unique up to an additive constant and such that $-\nu\Delta\mathbf{v} + \nabla\mathcal{Q} = \bar{\mathbf{f}}$ in $\mathbf{H}^{-1}$. □

We now state the main theorem of this section.

Theorem 2.5.4. For all $(\mathbf{f}, \mathbf{g}, h) \in \mathbf{L}^2(\Omega_0) \times \mathcal{H}_{00}^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_{i,o})$ the equation (2.5.1) admits a unique solution $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$. This solution satisfies the estimate

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega_0)} + \|p\|_{H^1(\Omega_0)} \leq C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_0)} + \|\mathbf{g}\|_{\mathcal{H}_{00}^{3/2}(\Gamma_0)} + \|h\|_{H^{1/2}(\Gamma_{i,o})}).$$

Proof. Let us work directly on the homogeneous system (2.5.8). We prove the existence of a unique pair $(\mathbf{v}, q) \in \mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ solution to this system. According to Theorems 2.5.2 and 2.5.3, $\mathbf{v}$ has to solve the variational formulation ($\star$). Hence we start with the solution of the variational formulation ($\star$) and we prove that it is the solution to (2.5.8). The plan is the following:

- *Step 1:* We extend the variational formulation ($\star$) on a larger domain $\Omega_{0,e}$ with a solution denoted by $\mathbf{v}_e$.
- *Step 2:* We prove that the solution $\mathbf{v}_e$ to this new variational formulation is in $\mathbf{H}^2$ in a neighbourhood of Γ_i.
- *Step 3:* We prove that the restriction of $\mathbf{v}_e$ to the initial domain Ω_0 is the solution $\mathbf{v}$ to ($\star$) which implies that $\mathbf{v}$ is $\mathbf{H}^2$ in a neighbourhood of Γ_i, and finally that $\mathbf{v} \in \mathbf{H}^2(\Omega_0)$.
- *Step 4:* We prove that all the pressures associated with $\mathbf{v}$ are in $H^1(\Omega_0)$ and are constant on $\Gamma_{i,o}$.

- *Step 5:* We conclude by taking the pressure satisfying $q = 0$ on $\Gamma_{i,o}$, so that the pair $(\mathbf{v}, q)$ is the unique solution to (2.5.8).

Step 1: Let η_e^0 be the function defined by

$$\eta_e^0 : \begin{cases} \eta^0(x) \ \text{ for all } x \in (0, L), \\ \eta^0(-x) \ \text{ for all } x \in (-L, 0). \end{cases}$$

We recall that η^0 is in $H^3(0, L)$ and that $\eta^0(0) = \eta_x^0(0) = 0$. Due to the even symmetry we have $\eta_e^0(0^-) = \eta_e^0(0^+) = 0$, $\eta_{e,x}^0(0^-) = \eta_{e,x}^0(0^+) = 0$, $\eta_{e,xx}^0(0^-) = \eta_{e,xx}^0(0^+)$ and thus we obtain $\eta_e^0 \in H^3(-L, L)$ and the curve $\Gamma_{0,e} = \{(x, y) \in \mathbb{R}^2 \mid x \in (-L, L),\, y = 1 + \eta_e^0(x)\}$ is $\mathcal{C}^2$. We set $\Omega_{0,e} = \{(x, y) \in \mathbb{R}^2 \mid x \in (-L, L),\, 0 < y < 1 + \eta_e^0(x)\}$.

Let $\mathbf{v}_e$ be the solution to

$$\nu \int_{\Omega_{0,e}} \nabla \mathbf{v}_e : \nabla \psi = \int_{\Omega_{0,e}} \bar{\mathbf{f}}_e \cdot \psi \ \text{ for all } \psi \in V_e,$$

where

$$V_e = \{\mathbf{v} \in \mathbf{H}^1(\Omega_{0,e}) \mid \text{div } \mathbf{v} = 0 \text{ in } \Omega_{0,e},\, \mathbf{v} = 0 \text{ on } \Gamma_{d,e},\, v_2 = 0 \text{ on } \Gamma_{i,o,e}\},$$
$$\Gamma_{d,e} = (-L, L) \times \{0\} \cup \Gamma_{0,e}, \ \ \Gamma_{i,e} = \{-L\} \times (0, 1), \ \ \Gamma_{i,o,e} = \Gamma_{i,e} \cup \Gamma_o,$$

and $\bar{\mathbf{f}}_e$ is the function defined by

$$\bar{\mathbf{f}}_e : \begin{cases} \bar{\mathbf{f}}_e = \bar{\mathbf{f}} \ \text{ in } \Omega_0, \\ \bar{\mathbf{f}}_e(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{\mathbf{f}}(-x, y) \ \text{ for all } (x, y) \in \Omega_{0,s}, \end{cases}$$

with $\Omega_{0,s} = \{(x, y) \in \mathbb{R}^2 \mid x \in (-L, 0),\, 0 < y < 1 + \eta_e^0(x)\}$.

Step 2: We use cutoff functions to prove the $\mathbf{H}^2$ regularity result near Γ_i. Let φ be a function in $\mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\varphi = 1$ on $\Omega_{\varphi,1}$ and $\underline{\text{support}(\varphi)} \subset \Omega_{\varphi,2}$, with $\Omega_{\varphi,1}$ and $\Omega_{\varphi,2}$ two open sets with smooth boundaries such that $\overline{\Omega_{\varphi,1}} \subset \overline{\Omega_{\varphi,2}} \subset \Omega_{0,e}$ and $\Omega_{\varphi,1}$ containing a neighbourhood of Γ_i.

Let $\mathcal{Q}_e$ be a pressure associated to $\mathbf{v}_e$. The pair $(\mathbf{v}_c, q_c) = (\varphi \mathbf{v}_e, \varphi \mathcal{Q}_e)$ satisfies, in $\mathbf{H}^{-1}(\Omega_{\varphi,2})$,

$$-\nu \Delta \mathbf{v}_c + \nabla q_c = -\nu \Delta \varphi \mathbf{v}_e - 2\nu \nabla \mathbf{v}_e \ \nabla \varphi + \mathcal{Q}_e \nabla \varphi + \varphi \bar{\mathbf{f}}_e.$$

Since $(\mathbf{v}_c, q_c)$ belongs to $\mathbf{H}_0^1(\Omega_{\varphi,2}) \times L^2(\Omega_{\varphi,2})$, the previous equality implies that $(\mathbf{v}_c, q_c)$ is a solution to the following Stokes equations (in the usual variational sense)

$$\begin{gathered} -\nu \Delta \mathbf{v}_c + \nabla q_c = -\nu \Delta \varphi \mathbf{v}_e - 2\nu \nabla \mathbf{v}_e \ \nabla \varphi + \mathcal{Q}_e \nabla \varphi + \varphi \bar{\mathbf{f}}_e \ \text{ in } \Omega_{\varphi,2}, \\ \text{div } \mathbf{v}_c = \mathbf{v}_e \cdot \nabla \varphi \ \text{ in } \Omega_{\varphi,2},\ \mathbf{v}_c = 0 \ \text{ on } \partial \Omega_{\varphi,2}. \end{gathered} \tag{2.5.9}$$

We then use known results for Stokes equations with Dirichlet boundary conditions (see for example [14, Chap IV, Theorem 5.8]) to obtain $(\mathbf{v}_c, q_c) \in \mathbf{H}^2(\Omega_{\varphi,2}) \times H^1(\Omega_{\varphi,2})$. As $(\mathbf{v}_c, q_c)$ is equal to $(\mathbf{v}_e, \mathcal{Q}_e)$ on $\Omega_{\varphi,1}$ we obtain the regularity result for $(\mathbf{v}_e, \mathcal{Q}_e)$ in a neighbourhood of Γ_i.

Step 3: We want to prove that the restriction to Ω_0 of $\mathbf{v}_e$ is the solution $\mathbf{v}$ to the variational formulation $(\star)$. Using the Lax-Milgram lemma we know that $\mathbf{v}_e$ satisfies

$$\frac{1}{2}\nu \int_{\Omega_{0,e}} |\nabla \mathbf{v}_e|^2 - \int_{\Omega_{0,e}} \bar{\mathbf{f}}_e \cdot \mathbf{v}_e = \min_{\boldsymbol{\varphi} \in V_e} \left(\frac{1}{2}\nu \int_{\Omega_{0,e}} |\nabla \boldsymbol{\varphi}|^2 - \int_{\Omega_{0,e}} \bar{\mathbf{f}}_e \cdot \boldsymbol{\varphi} \right). \tag{2.5.10}$$

Hence, using the symmetry properties of $\bar{\mathbf{f}}_e$ we can prove that the function $\mathbf{v}_s$ defined by

$$\mathbf{v}_s(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{v}_e(-x,y) \text{ for all } (x,y) \in \Omega_{0,e},$$

is also a solution to the minimization problem (2.5.10). As (2.5.10) admits a unique solution we obtain that $\mathbf{v}_s = \mathbf{v}_e$. The symmetry properties and the regularity of $\mathbf{v}_e$ imply that $v_{e,2} = 0$ on Γ_i. We can now prove that the restriction to Ω_0 of $\mathbf{v}_e$ is the solution $\mathbf{v}$ to $(\star)$. Let $\boldsymbol{\varphi}$ be a test function in V and denote by $\boldsymbol{\varphi}_e$ the function defined by

$$\boldsymbol{\varphi}_e : \begin{cases} \boldsymbol{\varphi}_e = \boldsymbol{\varphi} \text{ on } \Omega_0, \\ \boldsymbol{\varphi}_e(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{\varphi}(-x,y) \text{ for all } (x,y) \in \Omega_{0,s}. \end{cases}$$

Thanks to the condition $\varphi_2 = 0$ on $\Gamma_{i,o}$ we notice that $\boldsymbol{\varphi}_e$ is in $\mathbf{H}^1(\Omega_{0,e})$, and more precisely in V_e. Hence we can use $\boldsymbol{\varphi}_e$ as a test function in the variational formulation satisfied by $\mathbf{v}_e$, we obtain

$$\nu \int_{\Omega_{0,e}} \nabla \mathbf{v}_e : \nabla \boldsymbol{\varphi}_e = \int_{\Omega_{0,e}} \bar{\mathbf{f}}_e \cdot \boldsymbol{\varphi}_e.$$

Using the symmetry properties of $\mathbf{v}_e$, $\boldsymbol{\varphi}_e$ and $\bar{\mathbf{f}}_e$ we have

$$\int_{\Omega_{0,s}} \nabla \mathbf{v}_e : \nabla \boldsymbol{\varphi}_e = \int_{\Omega_0} \nabla \mathbf{v}_e : \nabla \boldsymbol{\varphi}_e,$$

and

$$\int_{\Omega_{0,s}} \bar{\mathbf{f}}_e \cdot \boldsymbol{\varphi}_e = \int_{\Omega_0} \bar{\mathbf{f}}_e \cdot \boldsymbol{\varphi}_e.$$

Hence,

$$\nu \int_{\Omega_0} \nabla \mathbf{v}_e : \nabla \boldsymbol{\varphi} = \int_{\Omega_0} \bar{\mathbf{f}} \cdot \boldsymbol{\varphi},$$

for all $\boldsymbol{\varphi}$ in V, which proves that the restriction to Ω_0 of $\mathbf{v}_e$ is the solution $\mathbf{v}$ to the variational formulation $(\star)$. Hence $\mathbf{v}$ is $\mathbf{H}^2$ in a neighbourhood of Γ_i. The same technique works for the boundary Γ_o which implies the regularity result on the whole domain Ω_0.

Step 4: Let $\mathcal{Q}$ be a pressure associated with $\mathbf{v}$. The regularity of $\mathbf{v}$ and the equality (in the sense of the distributions)

$$-\nu\Delta\mathbf{v} + \nabla\mathcal{Q} = \bar{\mathbf{f}},$$

imply that $\mathcal{Q}$ belongs to $H^1(\Omega_0)$. We now have to prove that $\mathcal{Q}$ is equal to a constant on $\Gamma_{i,o}$. Thanks to the regularity of $(\mathbf{v}, \mathcal{Q})$, the equality $-\Delta\mathbf{v} + \nabla\mathcal{Q} = \bar{\mathbf{f}}$ holds in $\mathbf{L}^2(\Omega_0)$. For all $\boldsymbol{\psi}$ in V we have

$$\int_{\Omega_0} \mathbf{f}\cdot\boldsymbol{\psi} = \int_{\Omega_0} (-\nu\Delta\mathbf{v} + \nabla\mathcal{Q})\cdot\boldsymbol{\psi} = \int_{\Omega_0} \nu\nabla\mathbf{v} : \nabla\boldsymbol{\psi} + \int_{\Gamma_{i,o}} \mathcal{Q}(\boldsymbol{\psi}\cdot\mathbf{n}),$$

and, using the definition of $\mathbf{v}$,

$$\int_{\Gamma_{i,o}} \mathcal{Q}(\boldsymbol{\psi}\cdot\mathbf{n}) = 0.$$

This implies that $\mathcal{Q}$ is constant on $\Gamma_{i,o}$. To see this, it is sufficient to prove that for all $\phi \in \mathcal{C}_c^\infty(\Gamma_{i,o})$ satisfying

$$\int_{\Gamma_{i,o}} \phi = 0,$$

there exists $\boldsymbol{\psi} \in V$ such that $\boldsymbol{\psi}\cdot\mathbf{n} = \phi$ on $\Gamma_{i,o}$. Let $\boldsymbol{\phi}$ be the function defined by

$$\boldsymbol{\phi} : \begin{cases} \boldsymbol{\phi} = 0 \text{ on } \Gamma_d, \\ \boldsymbol{\phi} = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \text{ on } \Gamma_{i,o}. \end{cases}$$

Using [27, Lemma 2.2] the equations

$$\begin{cases} \operatorname{div}\boldsymbol{\psi} = 0 & \Omega_0, \\ \boldsymbol{\psi} = \boldsymbol{\phi} & \Gamma_0, \end{cases}$$

admit a solution $\boldsymbol{\psi}$ in $\mathbf{H}^1(\Omega_0)$. Such a $\boldsymbol{\psi}$ belongs to V and satisfies $\boldsymbol{\psi}\cdot\mathbf{n} = \phi$ on $\Gamma_{i,o}$. Hence $\mathcal{Q}$ is constant on $\Gamma_{i,o}$.

Step 5: Among the pressures $\mathcal{Q}$ associated with $\mathbf{v}$ there exists a unique q in $H^1(\Omega_0)$ satisfying $q = 0$ in $\Gamma_{i,o}$ in the sense of the trace for Sobolev functions. The pair $(\mathbf{v}, q)$ in $\mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ is the unique solution to (2.5.8) and $(\mathbf{u}, p) = (\mathbf{v}, q) + (\mathbf{w}, H)$ is the unique solution to (2.5.1). The estimate on $(\mathbf{u}, p)$ follows from classical estimate for the Stokes equations (2.5.9) and Theorem 2.5.1 to estimate $\mathbf{w}$. □

According to Theorem 2.5.4 the Stokes operator A associated to (2.5.1) with homogeneous boundary condition is defined by

$$\mathcal{D}(A) = \mathbf{H}^2(\Omega_0) \cap V,$$

and for all $\mathbf{u} \in \mathcal{D}(A)$, $A\mathbf{u} = \nu\Pi\Delta\mathbf{u}$.

Theorem 2.5.5. The operator $(A, \mathcal{D}(A))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$. Moreover we have $\mathcal{D}(A^{1/2}) = V$.

Proof. The bilinear form associated with the operator A defined by

$$\forall (\mathbf{v}, \boldsymbol{\varphi}) \in V \times V, \ \ a(\mathbf{v}, \boldsymbol{\varphi}) = \nu \int_{\Omega_0} \nabla \mathbf{v} : \nabla \boldsymbol{\varphi},$$

is continuous and coercive, hence [10, Part 2, Theorem 2.2] proves that the operator A is the infinitesimal generator of an analytic semigroup. For the second part of the theorem we have, for all $\mathbf{u} \in \mathcal{D}(A)$,

$$\|\mathbf{u}\|_V = \langle -A\mathbf{u}, \mathbf{u}\rangle = \left\|(-A)^{1/2}\mathbf{u}\right\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)}.$$

By density, the previous equality is still true for $\mathbf{u} \in V$ which concludes the proof. $\square$

We now want to study (2.5.1) for weaker data using transposition method. The following lemma, used to solved non-zero divergence Stokes equations, is needed to obtain weak estimates on the pressure in Theorem 2.5.6.

Lemma 2.5.1. For all $\Phi \in H^1_0(\Omega_0)$ the system

$$\begin{aligned} &\operatorname{div} \mathbf{w} = \Phi \ \text{ in } \Omega_0, \\ &\mathbf{w} = 0 \ \text{ on } \Gamma_d, \ w_2 = 0 \ \text{ on } \Gamma_{i,o}, \end{aligned} \tag{2.5.11}$$

admits a solution $\mathbf{w} \in \mathbf{H}^2(\Omega_0)$ satisfying the estimate

$$\|\mathbf{w}\|_{\mathbf{H}^2(\Omega_0)} \le C \|\Phi\|_{H^1_0(\Omega_0)}.$$

Proof. If Φ has a zero average the result comes directly from [58, Chap II.2, Lemma 2.3.1]. This lemma gives the existence of a function $\mathbf{w} \in \mathbf{H}^2_0(\Omega_0)$ such that $\operatorname{div} \mathbf{w} = \Phi$. In the general case, the idea is to find a pair $(\mathbf{w}_0, \Phi_0)$ solution to (2.5.11), where Φ_0 has a non zero average, and to use it to come back to the previous framework.

Let $\delta > 0$ be the constant defined by (2.5.5) in Theorem 2.5.1 and $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a non zero non negative function compactly supported in $(0, \delta)$. Let $\theta \in \mathcal{C}^\infty(0, L)$ be such that $\theta = 0$ near 0 and $\theta = 1$ near L. Define $\mathbf{w}_0(x, y) = (\rho(y)\theta(x), 0)^T$ for all $(x, y) \in \Omega_0$. The function $\mathbf{w}_0$ is smooth and satisfies the boundary conditions in (2.5.11). Finally, set $\Phi_0(x, y) = \operatorname{div} \mathbf{w}_0(x, y) = \rho(y)\theta'(x)$ for all $(x, y) \in \Omega_0$ and remark that $\Phi_0 \in H^1_0(\Omega_0)$ and

$$\int_{\Omega_0} \rho(y)\theta'(x) dx dy = \int_0^\delta \rho(y) dy > 0.$$

We look for a solution to (2.5.11) under the form $\mathbf{w} = \widetilde{\mathbf{w}} + c\mathbf{w}_0$ with $c = \int_{\Omega_0} \Phi / \int_{\Omega_0} \Phi_0$. The function $\widetilde{\mathbf{w}}$ needs to satisfy

$$\begin{aligned} &\operatorname{div} \widetilde{\mathbf{w}} = \Phi - c\Phi_0 \ \text{ in } \Omega_0, \\ &\widetilde{\mathbf{w}} = 0 \ \text{ on } \Gamma_d, \ \widetilde{w}_2 = 0 \ \text{ on } \Gamma_{i,o}. \end{aligned}$$

The function $\widetilde{\Phi} = \Phi - c\Phi_0$ is in $H^1_0(\Omega_0)$ and has a zero average. The existence of $\widetilde{\mathbf{w}}$ follows from [58, Chap II.2, Lemma 2.3.1]. To prove the estimate on $\mathbf{w}$ remark that

$$c \leq \frac{\sqrt{\mu(\Omega_0)}}{\int_{\Omega_0} \Phi_0} \|\Phi\|_{H^1_0(\Omega_0)} .$$

□

Theorem 2.5.6. For all $(\mathbf{f}, \mathbf{g}, h) \in \mathbf{L}^2(\Omega_0) \times \mathcal{H}^{3/2}_{00}(\Gamma_0) \times H^{1/2}(\Gamma_{i,o})$ the solution $(\mathbf{u}, p)$ of the equation (2.5.1) satisfies the estimate

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega_0)} + \|p\|_{H^{-1}(\Omega_0)} \leq C(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega_0))'} + \|\mathbf{g}\|_{(\mathcal{H}^{1/2}(\Gamma_0))'} + \|h\|_{(H^{3/2}(\Gamma_{i,o}))'}). \tag{2.5.12}$$

Proof. The fluid part estimate is similar to [55, Lemma A.3] using as test function the solution $(\boldsymbol{\Psi}, \pi)$, given by Theorem 2.5.4, to

$$\begin{aligned} &-\nu\Delta\boldsymbol{\Psi} + \nabla\pi = \boldsymbol{\varphi}, \ \text{ div } \boldsymbol{\Psi} = 0 \ \text{ in } \Omega_0, \\ &\boldsymbol{\Psi} = 0 \ \text{ on } \Gamma_d, \ \ \Psi_2 = 0 \ \text{ and } \ \pi = 0 \ \text{ on } \Gamma_{i,o}, \end{aligned} \tag{2.5.13}$$

with $\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega_0)$. Let us prove the pressure estimate. For all $\Phi \in H^1_0(\Omega_0)$ consider the system

$$\begin{aligned} &-\nu\Delta\mathbf{v} + \nabla q = 0, \ \text{ div } \mathbf{v} = \Phi \ \text{ in } \Omega_0, \\ &\mathbf{v} = 0 \ \text{ on } \Gamma_d, \ \ v_2 = 0 \ \text{ and } \ q = 0 \ \text{ on } \Gamma_{i,o}. \end{aligned} \tag{2.5.14}$$

Using Lemma 2.5.1 and Theorem 2.5.4 this system admits a unique solution $(\mathbf{v}, q)$ in $\mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ which satisfies

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega_0)} + \|q\|_{H^1(\Omega_0)} \leq C \|\Phi\|_{H^1_0(\Omega_0)} .$$

Using Green's formula the following computations hold

$$\begin{aligned} 0 &= \int_{\Omega_0} (-\nu\Delta\mathbf{v} + \nabla q) \cdot \mathbf{u} \\ &= -\nu \int_{\Omega_0} \Delta\mathbf{u} \cdot \mathbf{v} - \nu \int_{\partial\Omega_0} \mathbf{u} \cdot (\nabla\mathbf{v}\,\mathbf{n}) + \nu \int_{\partial\Omega_0} \mathbf{v} \cdot (\nabla\mathbf{u}\,\mathbf{n}) + \int_{\partial\Omega_0} q(\mathbf{u} \cdot \mathbf{n}) \\ &= \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} - \int_{\Omega_0} \nabla p \cdot \mathbf{v} + \nu \int_{\partial\Omega_0} \mathbf{u} \cdot (\nabla\mathbf{v}\,\mathbf{n}) + \int_{\Gamma_0} q(\mathbf{u} \cdot \mathbf{n}) \\ &= \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} + \int_{\Omega_0} p\,\Phi - \int_{\Gamma_{i,o}} h(\mathbf{v} \cdot \mathbf{n}) + \nu \int_{\partial\Omega_0} \mathbf{u} \cdot (\nabla\mathbf{v}\,\mathbf{n}) + \int_{\Gamma_0} q(\mathbf{g} \cdot \mathbf{n}), \end{aligned}$$

and

$$\begin{aligned} \int_{\partial\Omega_0} \mathbf{u} \cdot (\nabla\mathbf{v}\,\mathbf{n}) &= \int_{\Gamma_0} \mathbf{g} \cdot (\nabla\mathbf{v}\,\mathbf{n}) + \int_{\Gamma_{i,o}} \mathbf{u} \cdot (\nabla\mathbf{v}\,\mathbf{n}) \\ &= \int_{\Gamma_0} g\, P_2(\nabla\mathbf{v}\,\mathbf{n}) + \int_{\Gamma_{i,o}} u_1 \partial_1 v_1, \end{aligned}$$

where $\mathbf{g} = (0, g)^T$ and P_2 is the vectorial projection on the second component. As $\partial_1 v_1 + \partial_2 v_2 = \Phi$ and $v_2 = 0$ on $\Gamma_{i,o}$ we notice that $\partial_1 v_1 = \Phi$ on $\Gamma_{i,o}$ and as $\Phi \in H_0^1(\Omega_0)$ we obtain $\partial_1 v_1 = 0$ on $\Gamma_{i,o}$. Finally

$$\begin{aligned}\left|\int_{\Omega_0} p\,\Phi\right| \leq\ & C(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega_0))'}\,\|\mathbf{v}\|_{\mathbf{H}^2(\Omega_0)} + \|h\|_{(H^{3/2}(\Gamma_{i,o}))'}\,\|\mathbf{v}\cdot\mathbf{n}\|_{H^{3/2}(\Gamma_{i,o})}\\ &+ \|\mathbf{g}\|_{(\mathcal{H}^{1/2}(\Gamma_0))'}\,\|P_2(\nabla\mathbf{v}\,\mathbf{n})\|_{\mathcal{H}^{1/2}(\Gamma_0)} + \|\mathbf{g}\|_{(\mathcal{H}^{1/2}(\Gamma_0))'}\,\|P_2(q\mathbf{n})\|_{\mathcal{H}^{1/2}(\Gamma_0)}),\\ \leq\ & C(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega_0))'} + \|\mathbf{g}\|_{(\mathcal{H}^{1/2}(\Gamma_0))'} + \|h\|_{(H^{3/2}(\Gamma_{i,o}))'})\,\|\Phi\|_{H_0^1(\Omega_0)},\end{aligned}$$

which implies the pressure estimate. □

As for [55, Theorem A.1] we now define a notion of weak solutions for (2.5.1). For $(\mathbf{f}, \mathbf{g}, h)$ in $(\mathbf{H}^2(\Omega_0))' \times (\mathcal{H}^{1/2}(\Gamma_0))' \times (H^{3/2}(\Gamma_{i,o}))'$ consider the following variational formulation:

Find $(\mathbf{u}, p) \in \mathbf{L}^2(\Omega_0) \times H^{-1}(\Omega_0)$ such that

$$\begin{aligned}\int_{\Omega_0} \mathbf{u}\cdot\boldsymbol{\varphi} =\ & \langle \mathbf{f}, \boldsymbol{\Psi}\rangle_{(\mathbf{H}^2(\Omega_0))',\mathbf{H}^2(\Omega_0)} + \langle h, \boldsymbol{\Psi}\cdot\mathbf{n}\rangle_{(H^{3/2}(\Gamma_{i,o}))',H^{3/2}(\Gamma_{i,o})}\\ &-\langle \mathbf{g}, P_2(\nabla\boldsymbol{\Psi}\,\mathbf{n})\rangle_{(\mathcal{H}^{1/2}(\Gamma_0))',\mathcal{H}^{1/2}(\Gamma_0)} + \langle \mathbf{g}, P_2(\pi\mathbf{n})\rangle_{(\mathcal{H}^{1/2}(\Gamma_0))',\mathcal{H}^{1/2}(\Gamma_0)},\end{aligned} \tag{2.5.15}$$

for all $\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega_0)$ and $(\boldsymbol{\Psi}, \pi)$ solution of (2.5.13), and

(2.5.16)

$$\begin{aligned}\langle p, \Phi\rangle_{H^{-1}(\Omega_0),H_0^1(\Omega_0)} =\ & -\langle \mathbf{f}, \mathbf{v}\rangle_{(\mathbf{H}^2(\Omega_0))',\mathbf{H}^2(\Omega_0)} + \langle h, \mathbf{v}\cdot\mathbf{n}\rangle_{(H^{3/2}(\Gamma_{i,o}))',H^{3/2}(\Gamma_{i,o})}\\ &-\langle \mathbf{g}, P_2(\nabla\mathbf{v}\,\mathbf{n})\rangle_{(\mathcal{H}^{1/2}(\Gamma_0))',\mathcal{H}^{1/2}(\Gamma_0)} + \langle \mathbf{g}, P_2(q\mathbf{n})\rangle_{(\mathcal{H}^{1/2}(\Gamma_0))',\mathcal{H}^{1/2}(\Gamma_0)},\end{aligned}$$

for all $\Phi \in H_0^1(\Omega_0)$ and $(\mathbf{v}, q)$ solution of (2.5.14).

Theorem 2.5.7. For all $(\mathbf{f}, \mathbf{g}, h) \in (\mathbf{H}^2(\Omega_0))' \times (\mathcal{H}^{1/2}(\Gamma_0))' \times (H^{3/2}(\Gamma_{i,o}))'$ there exists a unique solution $(\mathbf{u}, p) \in \mathbf{L}^2(\Omega_0) \times H^{-1}(\Omega_0)$ of (2.5.1) in the sense of the variational formulation (2.5.15)-(2.5.16). This solution satisfies the following estimate

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega_0)} + \|p\|_{H^{-1}(\Omega_0)} \leq C(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega_0))'} + \|\mathbf{g}\|_{(\mathcal{H}^{1/2}(\Gamma_0))'} + \|h\|_{(H^{3/2}(\Gamma_{i,o}))'}). \tag{2.5.17}$$

Proof. See [55, Theorem A.1]. □

2.5.2 Unsteady Stokes equations

Consider the unsteady Stokes equations

$$\begin{aligned}&\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f},\ \ \mathrm{div}\ \mathbf{u} = 0\ \text{ in } Q_T,\\ &\mathbf{u} = \mathbf{g}\ \text{ on } \Sigma_T^0,\ \mathbf{u} = 0\ \text{ on } \Sigma_T^b,\\ &u_2 = 0\ \text{ and }\ p = 0\ \text{ on } \Sigma_T^{i,o},\\ &\mathbf{u}(0) = \mathbf{u}^0\ \text{ on } \Omega_0.\end{aligned} \tag{2.5.18}$$

As for the steady Stokes equations, a nonhomogeneous boundary condition on the pressure $p = h$ in (2.5.18) can be handled directly with a lifting, hence through this section we assume that $h = 0$. We prove the existence and uniqueness of a solution to (2.5.18) in Theorem 2.5.8. Then we transform (2.5.18) to prove existence uniqueness and regularity result when the Dirichlet boundary condition $\mathbf{g}$ is less regular (see Theorem 2.5.9). We use this result to prove Lemma 2.3.2. Finally we specify the regularity result used in the study of the fluid structure system in Theorem 2.5.11 and we apply this result in Lemma 2.5.3.

Writing the equations satisfied by $\mathbf{u} - D\mathbf{g}$ and using standard semigroup techniques we obtain the following theorem. Remark that the assumption $\mathbf{u}^0 - D\mathbf{g}(0) \in V$ is equivalent to $\mathbf{u}^0 \in \mathbf{V}^1(\Omega_0)$, $\mathbf{u}^0 = \mathbf{g}$ on Γ_0 and $u_2^0 = 0$ on $\Gamma_{i,o}$.

Theorem 2.5.8. For all $\mathbf{g} \in L^2(0,T;\mathcal{H}^{3/2}_{00}(\Gamma_0)) \cap H^1(0,T;(\mathcal{H}^{1/2}(\Gamma_0))')$, $\mathbf{f} \in \mathbf{L}^2(Q_T)$ and $\mathbf{u}^0 \in \mathbf{H}^1(\Omega_0)$ satisfying the compatibility condition $\mathbf{u}^0 - D\mathbf{g}(0)$ belongs to V, the equation (2.5.18) admits a unique solution $(\mathbf{u}, p) \in \mathbf{H}^{2,1}(Q_T) \times L^2(0,T;H^1(\Omega_0))$. This solution satisfies the following estimate

$$
\begin{aligned}
&\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)} + \|p\|_{L^2(0,T;H^1(\Omega_0))} \\
&\le C(\|\mathbf{u}^0\|_{\mathbf{H}^1(\Omega_0)} + \|\mathbf{g}\|_{L^2(0,T;\mathcal{H}^{3/2}_{00}(\Gamma_0))} + \|\mathbf{g}'\|_{L^2(0,T;(\mathcal{H}^{1/2}(\Gamma_0)))'} + \|\mathbf{f}\|_{\mathbf{L}^2(Q_T)}).
\end{aligned}
$$

We now want to study (2.5.18) for $\mathbf{g} \in L^2(0,T;\mathcal{L}^2(\Gamma_0))$. We follow the approach of [55]. The operator A, using extrapolation method, can be extended to an unbounded operator $\widetilde{A}$ defined on $(\mathcal{D}(A^*))'$ with domain $\mathcal{D}(\widetilde{A}) = \mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$.

Definition 2.5.1. A function $\mathbf{u} \in \mathbf{L}^2(Q_T)$ is called a weak solution to (2.5.18) if $\Pi\mathbf{u}$ is a weak solution to the evolution equation

$$
\Pi\mathbf{u}' = \widetilde{A}\Pi\mathbf{u} + (-\widetilde{A})\Pi D\mathbf{g} + \Pi\mathbf{f}, \ \Pi\mathbf{u}(0) = \Pi\mathbf{u}^0, \tag{2.5.19}
$$

and $(\mathbb{I} - \Pi)\mathbf{u}$ is given by

$$
(\mathbb{I} - \Pi)\mathbf{u} = (\mathbb{I} - \Pi)D\mathbf{g} \ \text{ in } \mathbf{L}^2(Q_T). \tag{2.5.20}
$$

Remark that $A = A^*$ (the operator A is symmetric and onto from $\mathcal{D}(A)$ into $\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$). By definition to a weak solution for (2.5.19) (see [10]), $\Pi\mathbf{u} \in L^2(0,T,\mathbf{V}^0_{n,\Gamma_d}(\Omega_0))$ is solution to (2.5.19) if and only if for all $\Phi \in \mathcal{D}(A^*) = \mathcal{D}(A)$ the map $t \mapsto \int_{\Omega_0} \Pi\mathbf{u} \cdot \Phi$ belongs to $H^1(0,T)$ and

$$
\frac{d}{dt}\int_{\Omega_0} \Pi\mathbf{u}\cdot\Phi = \langle\widetilde{A}\Pi\mathbf{u}, \Phi\rangle_{\mathcal{D}(A)',\mathcal{D}(A)} + \langle -\widetilde{A}\Pi D\mathbf{g}, \Phi\rangle_{\mathcal{D}(A)',\mathcal{D}(A)} + \langle\Pi\mathbf{f}, \Phi\rangle_{\mathcal{D}(A)',\mathcal{D}(A)}. \tag{2.5.21}
$$

Using Green formula we compute the adjoint of the operator D.

Lemma 2.5.2. For all $\mathbf{f} \in \mathbf{L}^2(\Omega_0)$ the adjoint operator D^* of D is defined by

$$D^*\mathbf{f} = (-\nu\nabla\mathbf{v} + q)\,\mathbf{n},$$

where $(\mathbf{v}, q) \in \mathbf{H}^2(\Omega_0) \times H^1(\Omega_0)$ is the solution to

$$\begin{aligned} &-\nu\Delta\mathbf{v} + \nabla q = \mathbf{f}, \ \text{ div } \mathbf{v} = 0 \ \text{ in } \Omega_0, \\ &\mathbf{v} = 0 \ \text{ on } \Gamma_d, \ v_2 = 0 \ \text{ and } q = 0 \ \text{ on } \Gamma_{i,o}. \end{aligned}$$

Using that $\tilde{A}^* = A$ on $\mathcal{D}(A)$ the variational formulation (2.5.21) becomes

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_0} \Pi\mathbf{u}\cdot\Phi &= \int_{\Omega_0} \Pi\mathbf{u}\cdot A\Phi + \int_{\Gamma_0} \mathbf{g}\cdot D^*(-A)\Phi + \int_{\Omega_0} \Pi\mathbf{f}\cdot\Phi \\ &= \int_{\Omega_0} \Pi\mathbf{u}\cdot A\Phi + \int_{\Gamma_0} \mathbf{g}\cdot(-\nu\nabla\Phi + q)\,\mathbf{n}, + \int_{\Omega_0} \Pi\mathbf{f}\cdot\Phi, \end{aligned}$$

with $\nabla q = \nu(\mathbb{I} - \Pi)\Delta\Phi$. The previous equality follows from the uniqueness of the stationary Stokes system and the identity $-\nu\Delta\Phi + \nu(\mathbb{I} - \Pi)\Delta\Phi = -A\Phi$. Finally $\Pi\mathbf{u}$ is a weak solution to 2.5.19 if and only if

$$\frac{d}{dt}\int_{\Omega_0} \Pi\mathbf{u}\cdot\Phi = \int_{\Omega_0} \Pi\mathbf{u}\cdot A\Phi + \int_{\Gamma_0} \mathbf{g}\cdot(-\nu\nabla\Phi + q)\,\mathbf{n} + \int_{\Omega_0} \Pi\mathbf{f}\cdot\Phi \ \text{ for all } \Phi \in \mathcal{D}(A). \tag{2.5.22}$$

We can now state a theorem analogue to [55, Theorem 2.3].

Theorem 2.5.9. For all $\Pi\mathbf{u}^0 \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$, $\mathbf{g} \in L^2(0, T; \mathcal{L}^2(\Gamma_0))$ and $\mathbf{f} \in \mathbf{L}^2(Q_T)$ the equation (2.5.18) admits a unique weak solution $\mathbf{u}$ in the sense Definition 2.5.1. This solution satisfies the following estimate

(2.5.23)

$$\begin{aligned} &\|\Pi\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{1/2-\varepsilon}_{n,\Gamma_d}(\Omega_0))} + \|\Pi\mathbf{u}\|_{H^{1/4-\varepsilon/2}(0,T;\mathbf{V}^0(\Omega_0))} + \|(\mathbb{I} - \Pi)\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{1/2}(\Omega_0))} \\ &\leq C\left(\|\Pi\mathbf{u}^0\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)} + \|\mathbf{g}\|_{L^2(0,T;\mathcal{L}^2(\Gamma_0))} + \|\Pi\mathbf{f}\|_{L^2(0,T;\mathbf{V}^0_{n,\Gamma_d}(\Omega_0))}\right), \ \text{ for all } \varepsilon > 0. \end{aligned}$$

Proof. See [55, Theorem 2.3]. □

As in [55] we can prove that for $\mathbf{g} \in L^2(0, T; \mathcal{H}^{3/2}_{00}(\Gamma_0)) \cap H^1(0, T; \mathcal{H}^{-1/2}(\Gamma_0))$ a function $\mathbf{u}$ is solution to (2.5.18) in the sense of Theorem 2.5.8 if and only if $\mathbf{u}$ is a weak solution to (2.5.18)(in the sense of Definition 2.5.1). The following theorem characterize the pressure.

Theorem 2.5.10. For all $\mathbf{g} \in L^2(0, T; \mathcal{H}^{3/2}_{00}(\Gamma_0)) \cap H^1(0, T; (\mathcal{H}^{1/2}(\Gamma_0))')$, $\mathbf{f} \in \mathbf{L}^2(Q_T)$ and $\mathbf{u}^0 \in \mathbf{H}^1(\Omega_0)$ satisfying the compatibility condition $\mathbf{u}^0 - D\mathbf{g}(0)$ belongs to V, a pair $(\mathbf{u}, p) \in \mathbf{H}^{2,1}(Q_T) \times L^2(0, T; H^1(\Omega_0))$ is solution of (2.5.18) if and only if

$$\begin{aligned} &\Pi\mathbf{u}' = A\Pi\mathbf{u} + (-A)\Pi D\mathbf{g} + \Pi\mathbf{f}, \ \ \mathbf{u}(0) = \mathbf{u}^0, \\ &(I - \Pi)\mathbf{u} = (I - \Pi)D\mathbf{g}, \ \ p = \rho - q_t + p_{\mathbf{f}}, \end{aligned}$$

where

- $q \in H^1(0,T;H^1(\Omega_0))$ is the solution to

$$(2.5.24) \qquad \Delta q = 0 \ \text{ in } Q_T, \ \ \rho = 0 \ \text{ on } \Sigma_T^{i,o}, \ \ \frac{\partial q}{\partial \mathbf{n}} = \mathbf{g}\cdot\mathbf{n} \ \text{ on } \Sigma_T^0, \ \ \frac{\partial q}{\partial \mathbf{n}} = 0 \ \text{ on } \Sigma_T^b.$$

- $\rho \in L^2(0,T;H^1(\Omega_0))$ is the solution to

$$(2.5.25) \qquad \Delta \rho = 0 \ \text{ in } Q_T, \ \ \rho = 0 \ \text{ on } \Sigma_T^{i,o}, \ \ \frac{\partial \rho}{\partial \mathbf{n}} = \nu\Delta\Pi\mathbf{u}\cdot\mathbf{n} \ \text{ on } \Sigma_T^d,$$

  where $\nu\Delta\Pi\mathbf{u}\cdot\mathbf{n}$ is in $L^2(0,T;H^{-1/2}(\Gamma_d))$ thanks to the divergence theorem.

- $p_{\mathbf{f}} \in L^2(0,T;H^1(\Omega_0))$ is given by the identity $(I-\Pi)\mathbf{f} = \nabla p_{\mathbf{f}}$.

Proof. Writing $\mathbf{u} = \Pi\mathbf{u} + (\mathbb{I}-\Pi)\mathbf{u}$ in Equation (2.5.18), we have

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \Pi\mathbf{u}_t + (\mathbb{I}-\Pi)\mathbf{u}_t - \nu\Delta\Pi\mathbf{u} - \nu\Delta(\mathbb{I}-\Pi)\mathbf{u} + \nabla p = 0.$$

By definition of $(\mathbb{I}-\Pi)$ there exists $q \in H^1_{\Gamma_{i,o}}(\Omega_0)$ such that $\nabla q = (\mathbb{I}-\Pi)\mathbf{u}$. Using the condition div $\mathbf{u} = 0$ and $(\mathbb{I}-\Pi)\mathbf{u} = (\mathbb{I}-\Pi)D\mathbf{g}$ we obtain that q is solution to (2.5.24). As $\mathbf{g} \in H^1(0,T;\mathcal{H}^{-1/2}(\Gamma_0))$ the function q belongs to $H^1(0,T;H^1(\Omega_0))$.

The function $\Pi\mathbf{u}$ is solution to the equation

$$\Pi\mathbf{u}_t - \nu\Delta\Pi\mathbf{u} + \nabla\rho = 0,$$

with $\rho = p - \nu\Delta q + q_t = p + q_t$. Taking the divergence of the previous equation and the normal trace on Γ_d (which is well defined as $\Delta\Pi\mathbf{u}$ is in $L^2(0,T;\mathbf{L}^2(\Omega_0))$ with a divergence equal to zero) we obtain (2.5.25) and $\rho \in L^2(0,T;H^1(\Omega_0))$. □

We conclude this section with a regularity result, coming from the interpolation of the regularity results stated in Theorem 2.5.8 and Theorem 2.5.9, and an application to the operator $\mathcal{A}_1$ defined in Section 2.3.3.

Theorem 2.5.11. For all $\mathbf{g} \in L^2(0,T;\mathcal{H}^1_0(\Gamma_0)) \cap H^{1/2}(0,T;\mathcal{L}^2(\Gamma_0))$, $\mathbf{f} = 0$ and $\Pi\mathbf{u}^0 = 0$, the solution $\mathbf{u}$ to (2.5.19)-(2.5.20) satisfies the estimate

$$\|\Pi\mathbf{u}\|_{\mathbf{H}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T)} \leq C(\|\mathbf{g}\|_{L^2(0,T;\mathcal{H}^1_0(\Gamma_0))} + \|\mathbf{g}\|_{H^{1/2}(0,T;\mathcal{L}^2(\Gamma_0))}), \ \text{for all } \varepsilon > 0.$$

Lemma 2.5.3. The operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{H}$.

Proof. The first part is to prove that the unbounded operator $(\widetilde{\mathcal{A}}_1, \mathcal{D}(\widetilde{\mathcal{A}}_1))$, defined by

$$\mathcal{D}(\widetilde{\mathcal{A}}_1) = \{(\Pi\mathbf{u}, \eta_1, \eta_2) \in \mathbf{V}^1_{n,\Gamma_d}(\Omega_0) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s) \mid \Pi\mathbf{u} - \Pi D_s(\eta_2) \in V\}$$

and

$$\widetilde{\mathcal{A}}_1 = \begin{pmatrix} A & 0 & (-A)\Pi D_s \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix},$$

is the infinitesimal generator of a strongly continuous semigroup on $V^{-1} \times H_s$. Here, V^{-1} is the dual of V endowed with the norm

$$\mathbf{v} \mapsto \Big(\Big\langle (-A)^{-1}\mathbf{v}, \mathbf{v} \Big\rangle_{V,V^{-1}} \Big)^{1/2}.$$

This proof is similar to [56, Theorem 3.5]. Then we consider the evolution equation

$$\text{(2.5.26)} \qquad \frac{d}{dt}\begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \widetilde{\mathcal{A}}_1 \begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad \begin{pmatrix} \Pi\mathbf{u}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} \Pi\mathbf{u}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}.$$

The solution to (2.5.26) can be found in two steps. First we determine (η_1, η_2) and then $\Pi\mathbf{u}$. We recall that $(A_s, \mathcal{D}(A_s))$ is the infinitesimal generator of an analytic semigroup on H_s (see [16]). Let $(\Pi\mathbf{u}^0, \eta_1^0, \eta_2^0)$ be in $V^{-1} \times H_s$. Using [10, Chap 3, Theorem 2.2] we obtain $\eta_1 \in H^{3,3/2}(\Sigma_T^s)$ and $\eta_2 \in H^{1,1/2}(\Sigma_T^s)$. Now let us assume that $(\Pi\mathbf{u}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}$. We have to solve

$$(\Pi\mathbf{u})' = A\Pi\mathbf{u} + (-A)\Pi D_s(\eta_2),\ \Pi\mathbf{u}(0) = \Pi\mathbf{u}^0.$$

We split this equation in two parts $\Pi\mathbf{u} = \Pi\mathbf{u}_1 + \Pi\mathbf{u}_2$ with

$$(\Pi\mathbf{u}_1)' = A\Pi\mathbf{u}_1 + (-A)\Pi D_s(\eta_2),\ \Pi\mathbf{u}_1(0) = 0,$$

and

$$(\Pi\mathbf{u}_2)' = A\Pi\mathbf{u}_2,\ \Pi\mathbf{u}(0) = \Pi\mathbf{u}^0.$$

Using Theorem 2.5.11 we remark that $\Pi\mathbf{u}_1 \in \mathbf{H}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_T)$. For $\Pi\mathbf{u}_2$, [10, Chap 3, Theorem 2.2] shows that $\Pi\mathbf{u}_2 \in L^2(0,T;V) \cap H^1(0,T;V^{-1})$. Interpolation result [39, Theorem 3.1] ensures that $\Pi\mathbf{u}_2 \in \mathcal{C}([0,T];\mathbf{V}^0_{n,\Gamma_d}(\Omega_0))$.

Hence $(\Pi\mathbf{u}, \eta_1, \eta_2) \in \mathcal{C}([0,T];\mathbf{H})$ and the restriction to the semigroup $(e^{t\widetilde{\mathcal{A}}_1})_{t\in\mathbb{R}^+}$ to $\mathbf{H}$ is a strongly continuous semigroup on $\mathbf{H}$. Finally we can verify that the infinitesimal generator associated with this restriction is exactly the operator $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$. $\square$

2.5.3 Elliptic equations for the projector Π

In this section we prove higher regularity result for an elliptic equation, which implies the regularity result on the projector Π given in Lemma 2.3.1.

Lemma 2.5.4. Let f be in $H^1(\Omega_0)$ such that $f = 0$ on $\Gamma_{i,o}$ and g be in $H^{3/2}_{00}(\Gamma_0)$. Then the elliptic equation

$$\begin{cases} \Delta\rho = f \ \text{ in } \Omega_0, \\ \dfrac{\partial\rho}{\partial\mathbf{n}} = g(1+(\eta^0)^2)^{-1/2} \ \text{ on } \Gamma_0 \text{ and } \dfrac{\partial\rho}{\partial\mathbf{n}} = 0 \ \text{ on } \Gamma_b, \\ \rho = 0 \ \text{ on } \Gamma_{i,o}, \end{cases} \tag{2.5.27}$$

admits a unique solution $\rho \in H^3(\Omega_0)$.

Proof. H^3 regularity far from the corners of Ω_0 is obtained through classical arguments. To prove the H^3 regularity at the corners, say along $x = 0$, we first perform a symmetry with respect to $x = 0$ (step 1) and then a change of variables to transport the PDE on $(-L, L) \times (0, 1)$ (step 2).

Step 1: Using the notations of step 1 in the proof of Theorem 2.5.4 for η^0_e, $\Gamma_{0,e}$, $\Omega_{0,s}$ and $\Omega_{0,e}$ we define f_e and g_e by

$$f_e : \begin{cases} f_e = f \text{ in } \Omega_0, \\ f_e(x,y) = -f(-x,y) \ \text{ in } \Omega_{0,s}, \end{cases} \qquad g_e : \begin{cases} g_e = g \text{ in } \Gamma_0, \\ g_e(x,y) = g(-x,y) \ \text{ in } \Gamma_{0,e} \setminus \Gamma_0. \end{cases}$$

Assumptions on f and g ensure that (f_e, g_e) is in $H^1(\Omega_{0,e}) \times \mathbf{H}^{3/2}(\Gamma_{0,e})$. Define ρ_e by

$$\rho_e : \begin{cases} \rho_e = \rho \text{ in } \Omega_0, \\ \rho_e = -\rho(-x,y) \ \text{ for all } (x,y) \in \Omega_{0,s}. \end{cases}$$

Then $\rho_e \in H^2(\Omega_{0,e})$ and satisfies

$$\begin{cases} \Delta\rho_e = f_e \ \text{ in } \Omega_0, \\ \dfrac{\partial\rho_e}{\partial\mathbf{n}} = g_e(1+(\eta^0_e)^2)^{-1/2} \ \text{ on } \Gamma_{0,e} \text{ and } \dfrac{\partial\rho_e}{\partial\mathbf{n}} = 0 \ \text{ on } (-L,L)\times\{0\}, \\ \rho_e = 0 \ \text{ on } \ (\{-L\}\times(0,1)) \cup \Gamma_o. \end{cases}$$

Step 2: Let $\Omega_e = (-L, L) \times (0, 1)$ and φ be the change of variables

$$\varphi : \begin{cases} \Omega_{0,e} \longrightarrow \Omega_e, \\ (x,y) \mapsto (x,z) = \left(x, \dfrac{y}{1+\eta^0_e(x)}\right). \end{cases}$$

As in Theorem 2.5.1 the function φ transports $H^3(\Omega_{0,e})$ to $H^3(\Omega_e)$. Hence it is sufficient to prove the H^3 regularity after transport. Let $\mathbf{J}_\varphi$ be the Jacobian matrix of φ. Setting $\widetilde{\rho_e} = \rho \circ \varphi^{-1}$, $\widetilde{f_e} = |\mathbf{J}_\varphi|^{-1} f_e \circ \varphi^{-1}$ and $\widetilde{g_e}(x,1) = g_e(x, 1+\eta^0_e(x))$ the function $\widetilde{\rho_e}$ is solution to

$$\begin{cases} \operatorname{div}(A\nabla\widetilde{\rho_e}) = \widetilde{f_e} \ \text{ in } \Omega_e, \\ A\nabla\widetilde{\rho}_e \cdot \mathbf{n} = \widetilde{g_e} \ \text{ on } (-L,L)\times\{1\} \text{ and } A(x,z)\nabla\widetilde{\rho}_e \cdot \mathbf{n} = 0 \ \text{ on } (-L,L)\times\{0\}, \\ \widetilde{\rho_e} = 0 \ \text{ on } \ (\{-L\}\times(0,1)) \cup \Gamma_o, \end{cases} \tag{2.5.28}$$

where the matrix $A = (A_{i,j})_{1\le i,j\le 2} = |\det(\mathbf{J}_\varphi)|^{-1}\mathbf{J}_\varphi\mathbf{J}_\varphi^T$ is uniformly positive definite symmetric with coefficients in $W^{1,\infty}\cap H^2$.

Step 3: Deriving (2.5.28) with respect to x shows that $\partial_x\widetilde{\rho_e}$ satisfies (with $\partial_1 = \partial_x$ and $\partial_2 = \partial_z$)

$$\operatorname{div}(A\nabla(\partial_x\widetilde{\rho_e})) = \partial_x\widetilde{f}_e - F(A,\widetilde{\rho_e}), \tag{2.5.29}$$

with

$$\begin{aligned}F(A,\widetilde{\rho_e}) = \ &(\partial_{11}A_{11})\,\partial_1\widetilde{\rho_e} + (\partial_1 A_{11})\,\partial_{11}\widetilde{\rho_e} + (\partial_{11}A_{12})\,\partial_2\widetilde{\rho_e} + (\partial_1 A_{12})\,\partial_{12}\widetilde{\rho_e}\\ &+ (\partial_{12}A_{21})\,\partial_1\widetilde{\rho_e} + (\partial_1 A_{21})\,\partial_{21}\widetilde{\rho_e} + (\partial_{12}A_{22})\,\partial_2\widetilde{\rho_e} + (\partial_1 A_{22})\,\partial_{22}\widetilde{\rho_e},\end{aligned}$$

in the sense of the distributions on Ω_e. From here on, we localize near $(0,1)$.

Step 4: We use a bootstrap argument. The first step is to find an L^∞ estimate on $\nabla\widetilde{\rho_e}$. In the right hand-side of (2.5.29) the least regular terms are under the form $(\partial_{11}A_{11})\,\partial_x\widetilde{\rho_e}$ or $(\partial_{12}A_{22})\,\partial_z\widetilde{\rho_e}$. Sobolev embeddings show that these terms are in L^r for all $1 < r < 2$. Moreover the Neumann boundary condition involves $\partial_x\widetilde{g_e} - (\partial_1 A_{21})\,\partial_x\widetilde{\rho_e} - (\partial_1 A_{22})\,\partial_z\widetilde{\rho_e}$ where the least regular terms are traces of $W^{1,r}$ functions. Using the results of [2] and [3] we obtain that $\partial_x\widetilde{\rho_e}$ is in $W^{2,r}$. Then the embeddings $W^{2,r}\subset W^{1,r^*}\subset L^\infty$ with $r^* = \frac{2r}{2-r} > 2$ show that the terms under the form $(\partial_{11}A_{11})\,\partial_x\widetilde{\rho_e}$ are in L^2 and $(\partial_1 A_{21})\,\partial_x\widetilde{\rho_e}$ is in $H^{1/2}$ (on the boundary). Moreover using the equation (2.5.28) we obtain that $\partial_{zz}\widetilde{\rho_e}$ is in L^{r^*} and thus $\partial_z\widetilde{\rho_e}\in W^{1,r^*}\subset L^\infty$. Finally the right hand-side is in L^2 and the Neumann boundary condition in $H^{1/2}$ and thus $\partial_x\widetilde{\rho_e}$ is H^2 near $(0,1)$. For the regularity with respect to z we can use the equation (2.5.28) and $\widetilde{\rho_e}$ is H^3 in a neighbourhood of $(0,0)$.

Step 5: The strategy applies for $(0,0)$. If we come back to the initial equation on the domain Ω_0 we have proved that ρ is H^3 near Γ_i. The same proof can be used for the regularity near Γ_o and finally $\rho\in H^3(\Omega_0)$. □

Chapter 3

Existence of time-periodic solutions to a fluid–structure system

3.1 Introduction

In this chapter we are interested in the existence of time-periodic solutions for a fluid–structure system involving the incompressible Navier–Stokes equations coupled with a damped Euler–Bernoulli beam equation located on a part of the fluid domain boundary. This system can be used to model the blood flow through human arteries and serves as a benchmark problem for FSI solvers in hemodynamics. When the system is driven by periodic source terms, related for example to the periodic heartbeat, we expect a periodic response of the system. In this article, we prove the existence of time-periodic solutions for the fluid–structure system subject to small periodic impulses on the inflow and outflow boundaries. The study of this fluid–structure model in a periodic framework seems to be new.

For $L > 0$ consider the domain Ω in $\mathbb{R}^2$ defined by $\Omega = (0, L) \times (0, 1)$. The different components of the boundary $\partial\Omega$ are denoted by: $\Gamma_i = \{0\} \times (0, 1)$, $\Gamma_o = \{L\} \times (0, 1)$, $\Gamma_b = (0, L) \times \{0\}$, $\Gamma_s = (0, L) \times \{1\}$ and $\Gamma_d = \Gamma_s \cup \Gamma_i \cup \Gamma_b$. Let $T > 0$ be a period of the system, the domain of the fluid at the time $0 \leq t \leq T$ is denoted by $\Omega_{\eta(t)}$ and depends on the displacement of the beam $\eta : \Gamma_s \times (0, T) \mapsto (-1, +\infty)$. More precisely

$$\Omega_{\eta(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\, 0 < y < 1 + \eta(x, t)\},$$
$$\Gamma_{\eta(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\, y = 1 + \eta(x, t)\}.$$

For space-time domain we use the notations

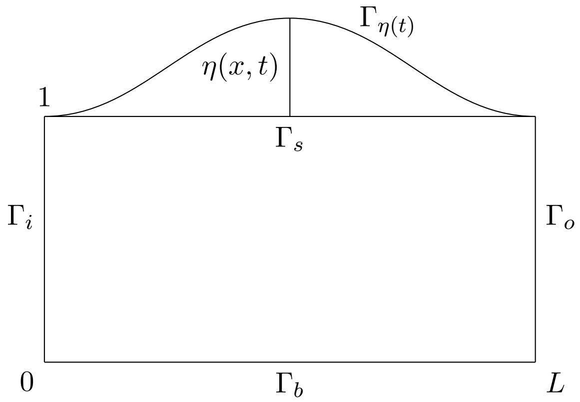

Figure 3.1: Fluid–structure system.

$$\Sigma_T^s = \Gamma_s \times (0,T),\ \Sigma_T^i = \Gamma_i \times (0,T),\ \Sigma_T^o = \Gamma_o \times (0,T),\ \Sigma_T^b = \Gamma_b \times (0,T),$$
$$\Sigma_T^d = \Gamma_d \times (0,T),\ \Sigma_T^\eta = \bigcup_{t\in(0,T)} \Gamma_{\eta(t)} \times \{t\},\ Q_T^\eta = \bigcup_{t\in(0,T)} \Omega_{\eta(t)} \times \{t\}.$$

Consider the T-periodic fluid–structure system

$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\,\sigma(\mathbf{u},p) = 0,\ \ \operatorname{div}\,\mathbf{u} = 0 \ \text{ in } Q_T^\eta,\\
&\mathbf{u} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma_T^\eta,\\
&\mathbf{u} = \boldsymbol{\omega}_1 \ \text{ on } \Sigma_T^i,\\
&u_2 = 0 \ \text{ and } \ p + (1/2)|\mathbf{u}|^2 = \omega_2 \ \text{ on } \Sigma_T^o,\\
&\mathbf{u} = 0 \ \text{ on } \Sigma_T^b,\ \ \mathbf{u}(0) = \mathbf{u}(T) \ \text{ in } \Omega_{\eta(0)},\\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = -J_{\eta(t)}\mathbf{e}_2 \cdot \sigma(\mathbf{u},p)_{|\Gamma_{\eta(t)}}\mathbf{n}_{\eta(t)} \ \text{ on } \Sigma_T^s,\\
&\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0,L\}\times(0,T),\\
&\eta(0) = \eta(T) \ \text{ and } \ \eta_t(0) = \eta_t(T) \ \text{ in } \Gamma_s,
\end{aligned} \tag{3.1.1}$$

where $\mathbf{u} = (u_1, u_2)$ is the fluid velocity, p the pressure, η the displacement of the beam and

$$\sigma(\mathbf{u},p) = -pI + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T),$$
$$\mathbf{n}_{\eta(t)} = J_{\eta(t)}^{-1}\begin{pmatrix} -\eta_x(x,t) \\ 1 \end{pmatrix},$$

with $J_{\eta(t)} = \sqrt{1+\eta_x^2}$. The constants $\beta \ge 0$, $\gamma > 0$, $\alpha > 0$ are parameters relative to the structure and $\nu > 0$ is the constant viscosity of the fluid. The periodic source terms $(\boldsymbol{\omega}_1, \omega_2)$ play the role of a 'pulsation' for the system and can model the heartbeat.

The fluid–structure system (3.1.1) has been investigated with different conditions on the inflow and outflow boundaries:

(DBC) homogeneous Dirichlet boundary conditions.

(PBC) periodic boundary conditions.

(PrBC) pressure boundary conditions.

For (DBC), the existence of strong solutions is proved in [9, 37, 56]. The first result, stated in [9], is the existence of local-in-time strong solutions for small data. This result is then improved in [56], where the stabilization process directly implies the existence of strong solutions, on an arbitrary time interval $[0,T]$ with $T>0$, for small data. Finally, in [37], the existence of strong solutions for small data and of local-in-time strong solutions without smallness assumptions on the initial data is proved. As specified in [15], the strategy developed in [37] works for zero (or small) initial beam displacement. This difficulty, purely nonlinear, was solved in [15] and more recently in [29].

For (PBC), the existence of global strong solutions without smallness assumptions on the initial data is proved in [28]. For a wide range of beam equations, depending on the positivity of the coefficients β,γ,α, the existence of local-in-time strong solutions without smallness assumptions is proved in [28, 29].

The third case (PrBC) is introduced in [49] where the existence of weak solutions is proved. We investigated, in Chapter 3, the existence of local-in-time strong solutions without smallness assumptions on the initial data, which includes non-small initial beam displacement, and the existence of strong solution on $[0,T]$ with $T>0$ for small data.

Here we are interested in the existence of time-periodic strong solutions. The term 'strong solutions' is related to the spacial regularity of the solution, which is typically, for the fluid, $\mathbf{H}^2$. In the semigroup terminology of evolution equations, the solutions considered in [9, 28, 29, 37, 56] correspond to strict solutions in L^2 (see Definition 3.4.1 in the appendix). Motivated by the stabilization of (3.1.1) in a neighbourhood of a periodic solution, we prove the existence of a time-periodic strict solution in $\mathcal{C}^0$ for (3.1.1) with Hölder regularity in time. Our result can be directly adapted for the boundary conditions (DBC)–(PBC)–(PrBC). The Dirichlet boundary condition on the inflow is motivated, once again, by stabilization purpose.

Let us describe the general strategy to construct a periodic solution for (3.1.1). First, we perform a change of variables mapping the moving domain $\Omega_{\eta(t)}$ into the fixed domain Ω. We then linearize and we rewrite the coupled system as an abstract evolution equation driven by an unbounded operator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ in Section 3.2. We prove that $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup and that its resolvent is compact. At this stage we use the abstract results developed in the appendix to ensure the existence of a time-periodic solution for the linear system. Finally, we study the nonlinear system in Section 3.3 with a fixed point argument in the space of periodic functions. The main theorem of this chapter, where the notation $\sharp$ denotes time-periodic functions, can be formulated as follows.

Theorem 3.1.1. Fix $\theta \in (0,1)$ and $T > 0$. There exists $R > 0$ such that, for all T-periodic source terms

$$(\boldsymbol{\omega}_1, \omega_2) \in \Big(\mathcal{C}^\theta_\sharp([0,T]; \mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; \mathbf{H}^{-1/2}(\Gamma_i))\Big) \times \mathcal{C}^\theta_\sharp([0,T]; H^{1/2}(\Gamma_o)),$$

satisfying

$$\|\boldsymbol{\omega}_1\|_{\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i))\cap\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o))} \leq R,$$

the system (3.1.1) admits a T-periodic strict solution $(\mathbf{u}, p, \eta)$ belonging to (after a change of variables mapping $\Omega_{\eta(t)}$ into Ω)

- $\mathbf{u} \in \mathcal{C}^\theta_\sharp([0,T]; \mathbf{H}^2(\Omega)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; \mathbf{L}^2(\Omega))$.
- $p \in \mathcal{C}^\theta_\sharp([0,T]; H^1(\Omega))$.
- $\eta \in \mathcal{C}^\theta_\sharp([0,T]; H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; H^2_0(\Gamma_s)) \cap \mathcal{C}^{2+\theta}_\sharp([0,T]; L^2(\Gamma_s))$.

The functional spaces are introduced in Section 3.1.2. In the appendix we present existence results for time-periodic abstract evolution equation. For a periodic evolution equation

$$\begin{cases} y'(t) = Ay(t) + f(t), \text{ for } t \in [0,T], \\ y(0) = y(T), \end{cases}$$

with $T > 0$, the existence of a solution is related to the spectral criteria $1 \in \rho(S(T))$ where $(S(t))_{t\geq 0}$ is the semigroup associated with A. This simple criteria follows from the Duhamel formula and is well known. It is stated, for example, in [20, 19] for T-periodic mild solutions and in [40, 41] for strict solutions in $\mathcal{C}^0$ with Hölder regularity in time (and for time-dependent operator $A(t)$). Our approach, however, specifies the different regularities that we can expect on the periodic solution, depending on the source term f. We also provide explicit conditions on the pair (A,T) to ensure that the spectral criteria is satisfied. Remark that the previous results always assume that A is the infinitesimal generator of an analytic semigroup. For abstract periodic evolution equations with weaker assumptions on A we refer to [13].

Let us conclude this introduction with a brief history on the existence of time-periodic solutions for the Navier–Stokes equations. This question was initially considered in 1960s in [31, 53, 54, 57]. In particular, in [31, 53, 54], the authors obtained a periodic weak solution by considering a fixed point of the Poincaré map which takes an initial value and provides the state of the corresponding initial-value problem at time T. The existence of strong solutions for small data is proved in [32] in 3D and without size rectriction in [59] in 2D. For more recent results with non-homogeneous boundary conditions see [33, 48]. The existence theory for the periodic Navier–Stokes equations in bounded domain is now as developed as the existence theory for the initial value problem. For unbounded domain the question is still delicate and was investigated, with zero boundary conditions at the infinity, in [25, 26, 35, 45, 46, 61]. For further references on the existence of periodic solutions for the Navier–Stokes equations we refer to [36].

The method developed in this chapter corresponds to the Poincaré map approach, applied on the whole coupled fluid–structure system. Note that the periodic solution obtained for the Navier–Stokes equations is usually unique. Here the free boundary makes the analysis of the uniqueness more complicated. For instance, we cannot considered the difference of two periodic solutions in their respective time-dependent domains, which may be different. The difference has to be taken after a change of variables mapping both periodic solutions in the same domain. In that case, energy estimates are difficult to obtain due to the higher order 'geometrical' nonlinear terms. The uniqueness question remains an open question in our work.

3.1.1 Equivalent system in a reference configuration

To fix the domain we perform the following change of variables

$$\mathcal{T}_0(t) : \begin{cases} \Omega_{\eta(t)} & \longrightarrow \Omega, \\ (x, y) & \longmapsto (x, z) = \left(x, \frac{y}{1+\eta(x,t)}\right). \end{cases} \tag{3.1.2}$$

Setting $Q_T = \Omega \times (0, T)$, $\hat{\mathbf{u}}(x, z, t) = \mathbf{u}(\mathcal{T}_0^{-1}(t)(x, z), t)$ and $\hat{p}(x, z, t) = p(\mathcal{T}_0^{-1}(t)(x, z), t)$, the system (3.1.1) becomes

$$\begin{aligned} &\hat{\mathbf{u}}_t - \nu\Delta\hat{\mathbf{u}} + \nabla\hat{p} = \mathbf{G}(\hat{\mathbf{u}}, \hat{p}, \eta), \ \ \text{div } \hat{\mathbf{u}} = \text{div } \mathbf{w}(\hat{\mathbf{u}}, \eta) \ \text{ in } Q_T, \\ &\hat{\mathbf{u}} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma_T^s, \\ &\hat{\mathbf{u}} = \boldsymbol{\omega}_1 \ \text{ on } \Sigma_T^i, \\ &\hat{u}_2 = 0 \ \text{ and } \ \hat{p} + (1/2)|\hat{\mathbf{u}}|^2 = \omega_2 \ \text{ on } \Sigma_T^o, \\ &\hat{\mathbf{u}} = 0 \ \text{ on } \Sigma_T^b, \ \ \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}(T) \ \text{ on } \Omega \\ &\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = \hat{p} - 2\nu\hat{u}_{2,z} + \Psi(\hat{\mathbf{u}}, \eta) \ \text{ on } \Sigma_T^s, \\ &\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0, L\} \times (0, T) \\ &\eta(0) = \eta(T) \ \text{ and } \ \eta_t(0) = \eta_t(T) \ \text{ in } \Gamma_s, \end{aligned} \tag{3.1.3}$$

with

$$
\begin{aligned}
&\mathbf{G}(\hat{\mathbf{u}},\hat{p},\eta) = -\eta\hat{\mathbf{u}}_t + \left[z\eta_t + \nu z\left(\frac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right]\hat{\mathbf{u}}_z \\
&+\nu\left[-2z\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \frac{z^2\eta_x^2-\eta}{1+\eta}\hat{\mathbf{u}}_{zz}\right] + z\eta_x\hat{p}_z\mathbf{e}_1 \\
&- z\eta\hat{p}_x\mathbf{e}_1 - (1+\eta)\hat{u}_1\hat{\mathbf{u}}_x + (z\eta_x\hat{u}_1 - \hat{u}_2)\hat{\mathbf{u}}_z, \\
&\mathbf{w}[\hat{\mathbf{u}},\eta] = -\eta\hat{u}_1\mathbf{e}_1 + z\eta_x\hat{u}_1\mathbf{e}_2, \\
&\Psi(\hat{\mathbf{u}},\eta) = \nu\left(\frac{\eta_x}{1+\eta}\hat{u}_{1,z} + \eta_x\hat{u}_{2,x} - \frac{\eta_x^2 z - 2\eta}{1+\eta}\hat{u}_{2,z}\right).
\end{aligned}
$$

We first study the linear system associated with (3.1.3) and then use a fixed point technique to prove the existence for the nonlinear system.

3.1.2 Function spaces

To deal with the mixed boundary conditions introduce the spaces

$$\mathbf{V}^0_{n,\Gamma_d}(\Omega) = \{\mathbf{v}\in\mathbf{L}^2(\Omega) \mid \mathrm{div}\ \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_d\},$$

and the orthogonal decomposition of $\mathbf{L}^2(\Omega) = L^2(\Omega,\mathbb{R}^2)$

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_{n,\Gamma_d}(\Omega) \oplus \mathrm{grad}\ H^1_{\Gamma_o}(\Omega),$$

where $H^1_{\Gamma_o}(\Omega) = \{u\in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_o\}$. Let $\Pi : \mathbf{L}^2(\Omega) \to \mathbf{V}^0_{n,\Gamma_d}(\Omega)$ be the so-called Leray projector associated with this decomposition. If $\mathbf{u}$ belongs to $\mathbf{L}^2(\Omega)$ then $\Pi\mathbf{u} = \mathbf{u} - \nabla p_{\mathbf{u}} - \nabla q_{\mathbf{u}}$ where $p_{\mathbf{u}}$ and $q_{\mathbf{u}}$ are solutions to the following elliptic equations

$$
\begin{aligned}
&p_{\mathbf{u}}\in H^1_0(\Omega),\ \ \Delta p_{\mathbf{u}} = \mathrm{div}\ \mathbf{u}\in H^{-1}(\Omega), \\
&q_{\mathbf{u}}\in H^1_{\Gamma_o}(\Omega),\ \ \Delta q_{\mathbf{u}} = 0,\ \ \frac{\partial q_{\mathbf{u}}}{\partial\mathbf{n}} = (\mathbf{u}-\nabla p_{\mathbf{u}})\cdot\mathbf{n} \text{ on } \Gamma_d,\ \ q_{\mathbf{u}} = 0 \text{ on } \Gamma_o.
\end{aligned}
\tag{3.1.4}
$$

Throughout this chapter the functions and spaces with vector values are written with a bold typography. For example $\mathbf{H}^2(\Omega) = H^2(\Omega,\mathbb{R}^2)$. As in Chapter 2 and using the notations in [38, Theorem 11.7], we introduce the space $H^{3/2}_{00}(\Gamma_s) = [H^1_0(\Gamma_s), H^2_0(\Gamma_s)]_{1/2}$. This space is a strict subspace of $H^{3/2}_0(\Gamma_s) = H^{3/2}(\Gamma_s)\cap H^1_0(\Gamma_s)$. Odd and even symmetries preserve the H^k-regularity for functions in $H^k_0(\Gamma_s)$ with $k = 1, 2$, thus, by interpolation, the $H^{3/2}$-regularity is also preserved for functions in $H^{3/2}_{00}(\Gamma_s)$. This property is used in Chapter 2 to handle the pressure boundary condition.

For the boundary condition on the inflow, we use the results developed in [47] for elliptic equations in a dihedron. In our case, the angle between Γ_i and $\Gamma_{s,0}$ is equal to $\frac{\pi}{2}$. If

$\boldsymbol{\omega}$ (resp. $\mathbf{g}$) denotes the boundary condition on Γ_i (resp. Γ_s), the Laplace and Stokes equations possess solutions with H^2-regularity near $C_{0,1} = (0,1)$ provided that the data are regular enough and that the compatibility conditions $\boldsymbol{\omega}(C_{0,1}) = \mathbf{g}(C_{0,1})$ is satisfied. To ensure these conditions, the non-homogeneous boundary condition on Γ_i is chosen in $H_0^{3/2}(\Gamma_i)$. Consider the Stokes system

$$\begin{aligned} &-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div}\,\mathbf{u} = 0 \ \text{ in } \Omega, \\ &\mathbf{u} = 0 \ \text{ on } \Gamma_d, \ \ u_2 = 0 \ \text{ and } p = 0 \ \text{ on } \Gamma_o. \end{aligned} \tag{3.1.5}$$

The energy space associated with (3.1.5) is

$$V = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \operatorname{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} = 0 \text{ on } \Gamma_d,\ u_2 = 0 \text{ on } \Gamma_o\}.$$

The regularity result for (3.1.5) is similar to Theorem 2.5.4 and we define the Stokes operator $(A_s, \mathcal{D}(A_s))$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$ by

$$\mathcal{D}(A_s) = \mathbf{H}^2(\Omega) \cap V, \ \text{ and for all}\, \mathbf{u} \in \mathcal{D}(A_s),\ A_s\mathbf{u} = \nu\Pi\Delta\mathbf{u}.$$

We also introduce the space $\mathbf{V}^s(\Omega) = \{\mathbf{u} \in \mathbf{H}^s(\Omega) \mid \operatorname{div}\,\mathbf{u} = 0\}$ for $s \geq 0$. To describe the Dirichlet boundary condition on Γ_s set

$$\begin{aligned} &\mathcal{L}^2(\Gamma_s) = \{0\} \times L^2(\Gamma_s), \qquad &&\mathcal{H}^{3/2}_{00}(\Gamma_s) = \{0\} \times H^{3/2}_{00}(\Gamma_s), \\ &\mathcal{H}^\kappa(\Gamma_s) = \{0\} \times H^\kappa(\Gamma_s), &&\mathcal{H}^\kappa_0(\Gamma_s) = \{0\} \times H^\kappa_0(\Gamma_s) \ \text{ for } \kappa \geq 0. \end{aligned}$$

For $\kappa \geq 0$, the dual space of $\mathcal{H}^\kappa(\Gamma_s)$ with $\mathcal{L}^2(\Gamma_s)$ as pivot space is denoted by $(\mathcal{H}^\kappa(\Gamma_s))'$.

For space-time dependent functions we use the notations introduced in [39]:

$$\begin{aligned} &\mathbf{L}^2(Q_T) = L^2(0,T;\mathbf{L}^2(\Omega)), \ \ \mathbf{H}^{p,q}(Q_T) = L^2(0,T;\mathbf{H}^p(\Omega)) \cap H^q(0,T;\mathbf{L}^2(\Omega)),\ p,q \geq 0, \\ &L^2(\Sigma^s_T) = L^2(0,T;L^2(\Gamma_s)), \ \ H^{p,q}(\Sigma^s_T) = L^2(0,T;H^p(\Gamma_s)) \cap H^q(0,T;L^2(\Gamma_s)), \ \ p,q \geq 0. \end{aligned}$$

If X is a space of functions and $\rho \geq 0$ we set

$$\begin{aligned} &\mathcal{C}^\rho_\sharp([0,T];X) := \{v_{|[0,T]} \mid v \in \mathcal{C}^\rho(\mathbb{R};X) \text{ is } T\text{-periodic}\}, \\ &H^\rho_\sharp(0,T;X) := \{v_{|[0,T]} \mid v \in H^\rho_{\mathrm{loc}}(\mathbb{R};X) \text{ is } T\text{-periodic}\}. \end{aligned}$$

3.2 Linear system

3.2.1 Stokes system with non-homogeneous mixed boundary conditions

In this section we consider the Stokes system

$$\begin{aligned} &\lambda\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \ \ \operatorname{div}\,\mathbf{u} = 0 \text{ in } \Omega, \\ &\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_s, \ \ \mathbf{u} = \boldsymbol{\omega} \ \text{ on } \Gamma_i, \\ &u_2 = 0 \ \text{ and } p = 0 \text{ on } \Gamma_o, \ \ \mathbf{u} = 0 \ \text{ on } \Gamma_b, \end{aligned} \tag{3.2.1}$$

with $\lambda \in \mathbb{C}$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{g} \in \mathcal{H}^{3/2}_{00}(\Gamma_s)$ and $\boldsymbol{\omega} \in \mathbf{H}^{3/2}_0(\Gamma_i)$. The following lemmas provide suitable lifting of the non-homogeneous Dirichlet boundary conditions on Γ_s and Γ_i.

Lemma 3.2.1. There exists $\Phi_s \in \mathcal{L}(\mathcal{H}^{3/2}_{00}(\Gamma_s), \mathbf{H}^2(\Omega))$ such that, for all $\mathbf{g} \in \mathcal{H}^{3/2}_{00}(\Gamma_s)$, $\mathbf{w} = \Phi_s(\mathbf{g})$ satisfies

$$\begin{aligned} &\text{div } \mathbf{w} = 0 \text{ in } \Omega, \\ &\mathbf{w} = \mathbf{g} \text{ on } \Gamma_s, \ \mathbf{w} = 0 \text{ on } \Gamma_i \cup \Gamma_b, \ w_2 = 0 \text{ on } \Gamma_o. \end{aligned} \tag{3.2.2}$$

Proof. The idea to solve (3.2.2) is to use a Stokes system with Dirichlet boundary conditions on an extended domain. We set $\Omega_e = (0, 2L) \times (0, 1)$, $\Gamma_{s,e} = (0, 2L) \times \{1\}$, $\Gamma_{b,e} = (0, 2L) \times \{0\}$, $\Gamma_{o,e} = \{2L\} \times (0, 1)$ and

$$\hat{\mathbf{g}} : \begin{cases} \hat{\mathbf{g}} = \mathbf{g} \text{ on } (0, L) \times \{1\}, \\ \hat{\mathbf{g}}(x, 1) = -\mathbf{g}(2L - x, 1) \text{ for } x \in (L, 2L). \end{cases}$$

Thanks to the properties of the space $H^{3/2}_{00}(\Gamma_s)$ with respect to symmetries, the function $\hat{\mathbf{g}}$ is in $\mathcal{H}^{3/2}_{00}(\Gamma_{s,e})$. Moreover, it has a zero average by construction. Consider the Stokes system

$$\begin{aligned} &-\nu\Delta\mathbf{v} + \nabla q = 0, \ \ \text{div } \mathbf{v} = 0 \ \text{ in } \Omega_e, \\ &\mathbf{v} = \hat{\mathbf{g}} \ \text{ on } \Gamma_{s,e}, \ \mathbf{v} = 0 \ \text{ on } \partial\Omega_e \setminus \Gamma_{s,e}. \end{aligned} \tag{3.2.3}$$

This system admits a unique solution $(\mathbf{v}, q) \in \mathbf{H}^2(\Omega_e) \times H^1(\Omega_e)$ (see for example [47]; note that one could not find $\mathbf{w}$ directly by solving (3.2.3) on Ω, since $\mathbf{g}$ does not necessarily have a zero average on Γ_s, contrary to $\hat{\mathbf{g}}$ on $\Gamma_{s,e}$. We introduce the function

$$\mathbf{v}_s(x, y) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{v}(2L - x, y) \text{ for all } (x, y) \in \Omega_e.$$

The function $\mathbf{v}_s \in \mathbf{H}^2(\Omega_e)$ still satisfies

$$\begin{aligned} &\text{div } \mathbf{v}_s = 0 \text{ in } \Omega_e, \\ &\mathbf{v}_s = \hat{\mathbf{g}} \text{ on } \Gamma_{s,e}, \ \mathbf{v}_s = 0 \text{ on } \partial\Omega_e \setminus \Gamma_{s,e}, \end{aligned}$$

and $\hat{\mathbf{v}} := \frac{\mathbf{v}+\mathbf{v}_s}{2}$ verifies $\hat{v}_2(L, y) = 0$ for all $y \in (0, 1)$. The restriction to Ω of $\hat{\mathbf{v}}$ is solution to (3.2.2). The linearity of the mapping $\mathbf{g} \mapsto \mathbf{w}$ is obvious from the construction above, and its continuity (that is, an estimate $\|\mathbf{w}\|_{\mathbf{H}^2(\Omega)} \le C \|\mathbf{g}\|_{\mathcal{H}^{3/2}_{00}(\Gamma_s)}$) follows from the classical estimates for the Stokes system with Dirichlet boundary conditions. $\square$

Lemma 3.2.2. There exists $\Phi_i \in \mathcal{L}(\mathbf{H}^{3/2}_0(\Gamma_i), \mathbf{H}^2(\Omega))$ such that, for all $\boldsymbol{\omega} \in \mathbf{H}^{3/2}_0(\Gamma_i)$, $\mathbf{w} = \Phi_i(\boldsymbol{\omega})$ satisfies

$$\begin{aligned} &\text{div } \mathbf{w} = 0 \text{ in } \Omega, \\ &\mathbf{w} = \boldsymbol{\omega} \text{ on } \Gamma_i, \ \mathbf{w} = 0 \text{ on } \Gamma_s \cup \Gamma_b, \ w_2 = 0 \text{ on } \Gamma_o. \end{aligned} \tag{3.2.4}$$

Proof. Once again we construct $\mathbf{w}$ by solving a Stokes system with Dirichlet boundary conditions. First, we have to compensate the non-zero average of $\boldsymbol{\omega}\cdot\mathbf{n}$ on Γ_i. Consider the function $\boldsymbol{\omega}^- \in \mathcal{H}^{3/2}_{00}(\Gamma_s)$ defined by

$$\boldsymbol{\omega}^-(x) = -\frac{\varphi(x)}{\int_{\Gamma_s}\varphi}\left(\int_{\Gamma_i}\boldsymbol{\omega}\cdot\mathbf{n}\right)\mathbf{e}_2, \quad \forall x\in(0,L),$$

where $\varphi\in\mathcal{C}^\infty_0(\Gamma_s)$ satisfies $\int_{\Gamma_s}\varphi\neq 0$. Consider then the system

$$\begin{aligned}&-\nu\Delta\mathbf{v}+\nabla q=0,\ \ \mathrm{div}\ \mathbf{v}=0 \text{ in } \Omega,\\ &\mathbf{v}=\boldsymbol{\omega} \text{ on } \Gamma_i,\ \mathbf{v}=\boldsymbol{\omega}^- \text{ on } \Gamma_s,\ \mathbf{v}=0 \text{ on } \Gamma_b\cup\Gamma_o.\end{aligned}$$

Using [47], we obtain a solution $(\mathbf{v},q)\in\mathbf{H}^2(\Omega)\times H^1(\Omega)$ to this system. Finally $\mathbf{w}=\mathbf{v}-\Phi_s(\boldsymbol{\omega}^-)$ satisfies (3.2.4). Once again, the linearity of $\Phi_i:\boldsymbol{\omega}\mapsto\mathbf{w}$ is trivial by construction, and its continuity follows from the classical estimates for the Stokes equations with Dirichlet boundary conditions, and from the construction of $\boldsymbol{\omega}^-$. $\square$

We can now specify the regularity results for (3.2.1).

Theorem 3.2.1. For all $(\mathbf{f},\mathbf{g},\boldsymbol{\omega})\in\mathbf{L}^2(\Omega)\times\mathcal{H}^{3/2}_{00}(\Gamma_s)\times\mathbf{H}^{3/2}_0(\Gamma_i)$, (3.2.1) admits a unique solution $(\mathbf{u},p)\in\mathbf{H}^2(\Omega)\times H^1(\Omega)$ which satisfies

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}+\|p\|_{H^1(\Omega)}\le C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}+\|\mathbf{g}\|_{\mathcal{H}^{3/2}_{00}(\Gamma_s)}+\|\boldsymbol{\omega}\|_{\mathbf{H}^{3/2}_0(\Gamma_i)}).$$

Proof. Consider $\mathbf{v}=\mathbf{u}-\Phi_s(\mathbf{g})-\Phi_i(\boldsymbol{\omega})$. The pair $(\mathbf{v},p)$ is solution to

$$\begin{aligned}&\lambda\mathbf{v}-\nu\Delta\mathbf{v}+\nabla p=\hat{\mathbf{f}},\ \ \mathrm{div}\ \mathbf{v}=0 \text{ in } \Omega,\\ &\mathbf{v}=0 \ \text{ on } \Gamma_d,\ u_2=0\ \text{ and } p=0 \text{ on } \Gamma_o,\end{aligned}$$

with $\hat{\mathbf{f}}=\mathbf{f}+\nu\Delta\Phi_s(\mathbf{g})+\nu\Delta\Phi_i(\boldsymbol{\omega})-\lambda\Phi_s(\mathbf{g})-\lambda\Phi_i(\boldsymbol{\omega})\in\mathbf{L}^2(\Omega)$. The $\mathbf{H}^2$-regularity of $\mathbf{v}$ in a neighbourhood of Γ_i is well known for Stokes with homogeneous Dirichlet conditions. The lower order term $\lambda\mathbf{v}$ does not impact the regularity of the system and can be dealt with a bootstrap argument. The regularity on a neighbourhood of Γ_o is proved in Theorem 2.5.4. Hence, $(\mathbf{v},p)\in\mathbf{H}^2(\Omega)\times H^1(\Omega)$, and thus $(\mathbf{u},p)\in\mathbf{H}^2(\Omega)\times H^1(\Omega)$ with the desired estimates. $\square$

We introduce the lifting operators:

- $L\in\mathcal{L}(\mathcal{H}^{3/2}_{00}(\Gamma_s),\mathbf{H}^2(\Omega)\times H^1(\Omega))$ defined by

  $$L(\mathbf{g})=(L_1(\mathbf{g}),L_2(\mathbf{g}))=(\mathbf{w}_1,\rho_1),$$

  where $(\mathbf{w}_1,\rho_1)$ is solution to (3.2.1) with $(\mathbf{f},\mathbf{g},\boldsymbol{\omega})=(\mathbf{0},\mathbf{g},\mathbf{0})$ and $\lambda=0$.

- $L_{\Gamma_i} \in \mathcal{L}(\mathbf{H}_0^{3/2}(\Gamma_i), \mathbf{H}^2(\Omega) \times H^1(\Omega))$ defined by

$$L_{\Gamma_i}(\boldsymbol{\omega}) = (L_{\Gamma_i,1}(\boldsymbol{\omega}), L_{\Gamma_i,2}(\boldsymbol{\omega})) = (\mathbf{w}_2, \rho_2),$$

  where $(\mathbf{w}_2, \rho_2)$ is the solution to (3.2.1) with $(\mathbf{f}, \mathbf{g}, \boldsymbol{\omega}) = (\mathbf{0}, \mathbf{0}, \boldsymbol{\omega})$ and $\lambda = 0$.
- $L_{\Gamma_o} \in \mathcal{L}(H^{1/2}(\Gamma_o), H^1(\Omega))$ a continuous lifting operator.

In order to express the pressure, we also need the operators:

- $N_s \in \mathcal{L}(\mathcal{H}_{00}^{3/2}(\Gamma_s), H^3(\Omega))$ defined by $N_s(\mathbf{g}) = p_1$ with

$$\begin{aligned} &\Delta p_1 = 0 \ \text{ in } \Omega, \\ &\frac{\partial p_1}{\partial \mathbf{n}} = \mathbf{g} \cdot \mathbf{n} \ \text{ on } \Gamma_s, \\ &\frac{\partial p_1}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_i \cup \Gamma_b, \\ &p_1 = 0 \ \text{ on } \Gamma_o. \end{aligned} \tag{3.2.5}$$

- $N_v \in \mathcal{L}(\mathbf{H}^2(\Omega), H^1(\Omega))$ defined by $N_v(\mathbf{u}) = p_2$ with

$$\begin{aligned} &\Delta p_2 = 0 \ \text{ in } \Omega, \\ &\frac{\partial p_2}{\partial \mathbf{n}} = \nu \Delta \Pi \mathbf{u} \cdot \mathbf{n} \ \text{ on } \Gamma_d, \\ &p_2 = 0 \ \text{ on } \Gamma_o. \end{aligned}$$

- $N_p \in \mathcal{L}(\mathbf{L}^2(\Omega), H^1_{\Gamma_o}(\Omega))$ defined by $N_p(\mathbf{f}) = p_3$ with $(I - \Pi)\mathbf{f} = \nabla p_3$.

Lemma 3.2.3. The operator N_s can be extended as follows:

- $N_s \in \mathcal{L}(\mathcal{H}^{3/2}(\Gamma_s)', L^2(\Omega))$.
- $N_s \in \mathcal{L}(\mathcal{H}^{1/2}(\Gamma_s)', H^1(\Omega))$.

Proof. The first result is obtained by duality. The second follows from interpolation techniques. $\square$

To prepare the matrix formulation of the fluid–structure system, we recast the Stokes system in terms of $\Pi \mathbf{u}$ and $(I - \Pi)\mathbf{u}$.

Theorem 3.2.2. Suppose that $\boldsymbol{\omega} = 0$ and $(\mathbf{f}, \mathbf{g}) \in \mathbf{L}^2(\Omega) \times \mathcal{H}_{00}^{3/2}(\Gamma_s)$. A pair $(\mathbf{u}, p)$ is solution to (3.2.1) if and only if

$$\begin{aligned} &\lambda \Pi \mathbf{u} - A_s \Pi \mathbf{u} + A_s \Pi L_1(\mathbf{g}) = \Pi \mathbf{f}, \\ &(I - \Pi)\mathbf{u} = \nabla N_s(\mathbf{g}), \\ &p = -\lambda N_s(\mathbf{g}) + N_v(\Pi \mathbf{u}) + N_p(\mathbf{f}). \end{aligned} \tag{3.2.6}$$

Proof. Remark that $\mathbf{u} - L_1(\mathbf{g})$ belongs to $\mathcal{D}(A_s)$ and

$$(3.2.7) \quad -\nu\Pi\Delta\mathbf{u} = -\nu\Pi\Delta(\mathbf{u} - L_1(\mathbf{g})) + \nu\Pi\Delta L_1(\mathbf{g}) \\ = -A_s\Pi(\mathbf{u} - L_1(\mathbf{g})) = -A_s\Pi\mathbf{u} + A_s\Pi L_1(\mathbf{g}).$$

In the previous identities we have used the extrapolation method to extend A_s as an unbounded operator in $\mathcal{D}(A_s^*)'$ with domain $\mathbf{V}^0_{n,\Gamma_d}(\Omega)$. Applying Π on the first line of (3.2.1) we obtain

$$\lambda\Pi\mathbf{u} - \nu\Pi\Delta\mathbf{u} = \Pi\mathbf{f},$$

which, using (3.2.7), provides the first line in (3.2.6). The second line follows directly from the elliptic equations (3.1.4) used to compute $(I - \Pi)\mathbf{u}$. Finally the pressure is obtained by applying $(I - \Pi)$ to the first line of (3.2.1). $\square$

3.2.2 Beam equation

Let $(A_{\alpha,\beta}, \mathcal{D}(A_{\alpha,\beta}))$ be the unbounded operator in $L^2(\Gamma_s)$ defined by $\mathcal{D}(A_{\alpha,\beta}) = H^4(\Gamma_s) \cap H^2_0(\Gamma_s)$ and, for all $\eta \in \mathcal{D}(A_{\alpha,\beta})$, $A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha\eta_{xxxx}$. The operator $A_{\alpha,\beta}$ is self-adjoint and is an isomorphism from $\mathcal{D}(A_{\alpha,\beta})$ to $L^2(\Gamma_s)$. It can be extended by duality as an isomorphism from $L^2(\Gamma_s)$ to $\mathcal{D}(A_{\alpha,\beta})'$ and, using interpolation, $A_{\alpha,\beta} \in \mathcal{L}(H^{7/2}(\Gamma_s) \cap H^2_0(\Gamma_s), H^{-1/2}(\Gamma_s))$.

The space $H^2_0(\Gamma_s)$ is equipped with the inner product

$$\langle \eta_1, k_1 \rangle_{H^2_0(\Gamma_s)} = \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1(-A_{\alpha,\beta})^{1/2}k_1.$$

The unbounded operator $(A_b, \mathcal{D}(A_b))$ associated with the beam, in $H_b = H^2_0(\Gamma_s) \times L^2(\Gamma_s)$, is defined by

$$\mathcal{D}(\mathcal{A}_b) = (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s) \text{ and } \mathcal{A}_b = \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} & \gamma\Delta_s \end{pmatrix}.$$

Theorem 3.2.3. The operator $(\mathcal{A}_b, \mathcal{D}(\mathcal{A}_b))$ is the infinitesimal generator of an analytic semigroup on H_b.

Proof. See [16]. $\square$

3.2.3 Semigroup formulation of the linear fluid–structure system

Consider a period $T > 0$. Set $\theta \in (0, 1)$ and

$$(\boldsymbol{\omega_1}, \omega_2) \in \left(\mathcal{C}^\theta_\sharp([0,T]; \mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T]; \mathbf{H}^{-1/2}(\Gamma_i))\right) \times \mathcal{C}^\theta_\sharp([0,T]; H^{1/2}(\Gamma_o)).$$

For $(\mathbf{f}, \Theta, h)$ in $\mathcal{C}^\theta_\sharp(0,T;\mathbf{L}^2(\Omega)) \times \mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o)) \times \mathcal{C}^\theta_\sharp([0,T];L^2(\Gamma_s))$ and

$$\mathbf{w} \in \mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{L}^2(\Omega)) \cap \mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)),$$

consider the following linear system

$$\begin{aligned}
&\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \ \ \text{div}\ \mathbf{u} = \text{div}\ \mathbf{w} \ \text{ in } Q_T \\
&\mathbf{u} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma^s_T, \ \ \mathbf{u} = \boldsymbol{\omega}_1 \ \text{ on } \Sigma^i_T, \\
&u_2 = 0 \text{ and } p = \omega_2 + \Theta \ \text{ on } \Sigma^o_T, \\
&\mathbf{u} = 0 \ \text{ on } \Sigma^b_T, \ \ \mathbf{u}(0) = \mathbf{u}(T) \ \text{ in } \Omega, \\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = p - 2\nu u_{2,z} + h \ \text{ in } \Sigma^s_T, \\
&\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0,L\} \times (0,T), \\
&\eta(0) = \eta(T) \ \text{ and } \ \eta_t(0) = \eta_t(T) \ \text{ in } \Gamma_s.
\end{aligned} \tag{3.2.8}$$

For a scalar function η defined on Γ_s we use the notation $L_1(\eta) = L_1(\eta\mathbf{e}_2)$. We look for a solution to (3.2.8) under the form $(\mathbf{u}, p, \eta) = (\mathbf{v}, q, \eta) + (\mathbf{w} + L_{\Gamma_i,1}(\boldsymbol{\omega}_1), L_{\Gamma_o}(\omega_2) + L_{\Gamma_o}(\Theta) + L_{\Gamma_i,2}(\boldsymbol{\omega}_1), 0)$ with $(\mathbf{v}, q, \eta)$ solution to

$$\begin{aligned}
&\mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla q = F, \ \ \text{div}\ \mathbf{v} = 0 \ \text{ in } Q_T, \\
&\mathbf{v} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma^s_T, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma^i_T, \\
&v_2 = 0 \text{ and } q = 0 \ \text{ on } \Sigma^o_T, \\
&\mathbf{v} = 0 \ \text{ on } \Sigma^b_T, \ \ \mathbf{v}(0) = \mathbf{v}(T) \ \text{ in } \Omega, \\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = q + H \ \text{ in } \Sigma^s_T, \\
&\eta = 0 \ \text{ and } \ \eta_x = 0 \ \text{ on } \{0,L\} \times (0,T), \\
&\eta(0) = \eta(T) \ \text{ and } \ \eta_t(0) = \eta_t(T) \ \text{ in } \Gamma_s,
\end{aligned} \tag{3.2.9}$$

where $F = \mathbf{f} - \mathbf{w}_t + \nu\Delta\mathbf{w} - \partial_t L_{\Gamma_i,1}(\boldsymbol{\omega}_1) - \nabla L_{\Gamma_o}(\omega_2) - \nabla L_{\Gamma_o}(\Theta)$ and $H = w_{2,z} + L_{\Gamma_i,2}(\boldsymbol{\omega}_1) + L_{\Gamma_o}(\omega_2) + L_{\Gamma_o}(\Theta) + h$.

Theorem 3.2.4. Suppose that $\eta_t \in \mathcal{C}^{1+\theta}_\sharp([0,T];L^2(\Gamma_s)) \cap \mathcal{C}^\theta_\sharp([0,T];H^2_0(\Gamma_s))$. A pair

$$(\mathbf{v}, q) \in \Big(\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{L}^2(\Omega)) \cap \mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^2(\Omega))\Big) \times \mathcal{C}^\theta_\sharp([0,T];H^1(\Omega)) \tag{3.2.10}$$

obeys the fluid equations of (3.2.9) if and only if

$$\begin{aligned}
&\Pi\mathbf{v}_t = A_s\Pi\mathbf{v} - A_s\Pi L_1(\eta_t), \ \ \mathbf{v}(0) = \mathbf{v}(T), \\
&(I - \Pi)\mathbf{v} = \nabla N_s(\eta_t), \\
&q = -N_s(\eta_t)_t + N_v(\Pi\mathbf{v}) + N_p(F).
\end{aligned} \tag{3.2.11}$$

Proof. A pair $(\mathbf{v}, q)$ as in (3.2.10) is solution to the fluid equations in (3.2.9) if and only if

$$\begin{aligned}
&-\nu\Delta\mathbf{v} + \nabla q = F - \mathbf{v}_t, \ \ \text{div}\ \mathbf{v} = 0 \text{ in } Q_T, \\
&\mathbf{v} = \eta_t \mathbf{e}_2 \ \text{ on } \Sigma^s_T, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma^i_T, \\
&v_2 = 0 \ \text{ and } q = 0 \text{ on } \Sigma^o_T, \ \ \mathbf{v} = 0 \ \text{ on } \Sigma^b_T.
\end{aligned}$$

Apply then Theorem 3.2.2 to conclude. □

Introduce the space

$$\mathbf{H} = \mathbf{V}^0_{n,\Gamma_d}(\Omega) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s),$$

equipped with the inner product

$$\langle (\mathbf{u}, \eta_1, \eta_2), (\mathbf{v}, \zeta_1, \zeta_2) \rangle_{\mathbf{H}} = \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{L}^2(\Omega)} + \langle \eta_1, \zeta_1 \rangle_{H^2_0(\Gamma_s)} + \langle \eta_2, \zeta_2 \rangle_{L^2(\Gamma_s)}.$$

Owing to Theorem 3.2.4, System (3.2.9) can be recast in terms of $(\Pi\mathbf{v}, \eta, \eta_t)$:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \Pi\mathbf{v} \\ \eta \\ \eta_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} \Pi\mathbf{v} \\ \eta \\ \eta_t \end{pmatrix} + \mathbf{F}, \quad \begin{pmatrix} \Pi\mathbf{v}(0) \\ \eta(0) \\ \eta_t(0) \end{pmatrix} = \begin{pmatrix} \Pi\mathbf{v}(T) \\ \eta(T) \\ \eta_t(T) \end{pmatrix}, \\ (I - \Pi)\mathbf{v} = \nabla N_s(\eta_t), \\ q = -N_s(\eta_t)_t + N_v(\Pi\mathbf{v}) + N_p(F), \end{cases} \tag{3.2.12}$$

where

$$\mathbf{F} = \begin{pmatrix} \Pi F \\ 0 \\ (I + N_s)^{-1}(N_p(F) + H) \end{pmatrix},$$

and $\mathcal{A}$ is the unbounded operator in $\mathbf{H}$ defined by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{ (\Pi\mathbf{v}, \eta_1, \eta_2) \in \mathbf{V}^2_{n,\Gamma_d}(\Omega) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s) \\ \big| \Pi\mathbf{v} - \Pi L_1(\eta_2) \in \mathcal{D}(A_s) \}, \end{aligned}$$

and

$$\mathcal{A} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I + N_s)^{-1} \end{pmatrix} \begin{pmatrix} A_s & 0 & -A_s \Pi L_1 \\ 0 & 0 & I \\ N_v & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}, \tag{3.2.13}$$

with $\Delta_s = \partial_{xx}$.

3.2.4 Analyticity of $\mathcal{A}$

The unbounded operator $\mathcal{A}$ has already been studied, with small variations related to the boundary conditions, in Chapter 2.

Theorem 3.2.5. The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$. Moreover, the resolvent of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is compact.

Proof. We write $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ with

$$\mathcal{A}_1 = \begin{pmatrix} A_s & 0 & -A_s \Pi L_1 \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (I+N_s)^{-1}N_v & K_s A_{\alpha,\beta} & K_s\delta\Delta_s \end{pmatrix},$$

where $K_s = (I+N_s)^{-1} - I$. We start with the resolvent of $\mathcal{A}_1$. Let $\lambda_b \in \mathbb{R}$ be such that $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \lambda_b\} \subset \rho(\mathcal{A}_b)$. For $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda \geq \lambda_b$, consider the system

$$\begin{aligned}
&\lambda\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = F_1, \ \ \mathrm{div}\ \mathbf{u} = 0 \ \text{ in } \Omega, \\
&\mathbf{u} = \eta_2\mathbf{e}_2 \ \text{ on } \Gamma_s, \ \ \mathbf{u} = 0 \ \text{ on } \Gamma_i, \\
&u_2 = 0 \text{ and } p = 0 \ \text{ on } \Gamma_o, \\
&\mathbf{u} = 0 \ \text{ on } \Gamma_b, \\
&\lambda\eta_1 - \eta_2 = F_2 \text{ on } \Gamma_s, \\
&\lambda\eta_2 - \beta\eta_{1,xx} - \gamma\eta_{2,xx} + \alpha\eta_{1,xxxx} = F_3 \ \text{ on } \Sigma_T^s, \\
&\eta_1 = 0 \ \text{ and } \ \eta_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,T),
\end{aligned} \tag{3.2.14}$$

for $(F_1, F_2, F_3) \in \mathbf{H}$. This system is triangular: the beam equation can be solved first, and its solution used then to solve the Stokes system. The assumption on λ ensures the existence of $(\eta_1, \eta_2) \in (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s)$ solution to the beam equations and such that

$$\|\eta_1\|_{H^4(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)} \leq C(\|F_2\|_{H_0^2(\Gamma_s)} + \|F_3\|_{L^2(\Gamma_s)}).$$

The Stokes system can then be solved, and we find $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ solution to $(3.2.14)_{1-4}$ such that

$$\begin{aligned}
\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} &\leq C(\|\eta_2\|_{H_0^2(\Gamma_s)} + \|F_1\|_{\mathbf{L}_2(\Omega)}) \\
&\leq C(\|F_1\|_{\mathbf{L}_2(\Omega)} + \|F_2\|_{H_0^2(\Gamma_s)} + \|F_3\|_{L^2(\Gamma_s)}).
\end{aligned}$$

System (3.2.14) is equivalent to

$$\begin{cases}
\lambda \begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_1 \begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \\
(I - \Pi)\mathbf{u} = \nabla N_s(\eta_2), \\
p = -\lambda N_s(\eta_2) + N_v(\Pi\mathbf{u}),
\end{cases}$$

and the reasoning above shows that $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \lambda_b\} \subset \rho(\mathcal{A}_1)$. The resolvent estimates on $\mathcal{A}_1$ are similar to Theorem 2.3.2 and $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}))$ is sectorial. Using a similar technique as in Lemma 2.5.3 we prove that $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{H}$. Finally, the previous properties imply that $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$.

As in Theorem 2.3.3, the term $\mathcal{A}_2$ is $\mathcal{A}_1$-bounded with relative bound zero. Using [52, Section 3.2, Theorem 2.1], we thus obtain the analyticity of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. The Rellich compact embedding theorem ensures that $\mathcal{D}(\mathcal{A}) \overset{c}{\hookrightarrow} \mathbf{H}$ and the resolvent of $\mathcal{A}$ is therefore compact. $\square$

3.2.5 Time-periodic solutions of the linear system

In this section we apply the existence results of periodic solutions developed in the appendix to the system (3.2.8).

In the appendix, we prove the existence of time-periodic solutions for abstract evolution equations $y'(t) = Ay(t) + f(t)$ under the assumption (3.4.4). This assumption is a restriction on the period T of the system depending on the eigenvalues of A lying on the imaginary axis. Here, this condition does not restrict the choice of T as we are able to prove that all the non-zero eigenvalues of $\mathcal{A}$ have a negative real part. Indeed, let $\lambda \in \mathbb{C}$ be a non-zero eigenvalues of $\mathcal{A}$ and $(\Pi\mathbf{u}, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A})$ be an associated eigenvector. The system

$$\lambda \begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} - \mathcal{A} \begin{pmatrix} \Pi\mathbf{u} \\ \eta_1 \\ \eta_2 \end{pmatrix} = 0,$$

is equivalent to

$$\begin{aligned}
&\lambda\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0, \ \ \text{div } \mathbf{u} = 0 \ \text{ in } \Omega, \\
&\mathbf{u} = \eta_2 \mathbf{e}_2 \ \text{ on } \Gamma_s, \ \ \mathbf{u} = 0 \ \text{ on } \Gamma_i, \\
&u_2 = 0 \text{ and } p = 0 \ \text{ on } \Gamma_o, \\
&\mathbf{u} = 0 \ \text{ on } \Gamma_b, \\
&\lambda\eta_1 - \eta_2 = 0 \text{ on } \Gamma_s, \\
&\lambda\eta_2 - \beta\eta_{1,xx} - \gamma\eta_{2,xx} + \alpha\eta_{1,xxxx} = p \ \text{ on } \Gamma_s, \\
&\eta_1 = 0 \ \text{ and } \ \eta_{1,x} = 0 \ \text{ on } \{0, L\},
\end{aligned} \tag{3.2.15}$$

with $\mathbf{u} = \Pi\mathbf{u} + \nabla N_s(\eta_2)$ and $p = -\lambda N_s(\eta_2) + N_v(\Pi\mathbf{u})$. Multiplying the first line of (3.2.15) by $\overline{\mathbf{u}}$ (the complex conjugate of $\mathbf{u}$) and integrating by part we obtain

$$\lambda \int_\omega |\mathbf{u}|^2 + \nu \int_\Omega |\nabla\mathbf{u}|^2 + \int_{\Gamma_s} p\overline{\eta}_2 = 0.$$

Then, multiplying the second line of the beam equation by $\overline{\eta}_2$, using the identity $\lambda\eta_1 = \eta_2$ and integration by part we obtain

$$\int_{\Gamma_s} p\overline{\eta}_2 = \lambda \int_{\Gamma_s} |\eta_2|^2 + \beta\overline{\lambda} \int_{\Gamma_s} |\eta_{1,x}|^2 + \gamma \int_{\Gamma_s} |\eta_{2,x}|^2 + \alpha\overline{\lambda} \int_{\Gamma_s} |\eta_{1,xx}|^2.$$

Combining the previous energy estimates we obtain

$$\lambda\left[\int_\Omega |\mathbf{u}|^2 + \int_{\Gamma_s} |\eta_2|^2\right] + \overline{\lambda}\left[\beta\int_{\Gamma_s} |\eta_{1,x}|^2 + \alpha\int_{\Gamma_s} |\eta_{1,xx}|^2\right] + \nu\int_\Omega |\nabla\mathbf{u}|^2 + \gamma\int_{\Gamma_s} |\eta_{2,x}|^2 = 0.$$

Taking the real part of the previous identity we deduce that Re $\lambda < 0$. It is easily verified that $0 \notin \sigma_p(\mathcal{A})$ (recall that $\sigma_p(\mathcal{A}) = \sigma(\mathcal{A})$ as the resolvent of $\mathcal{A}$ is compact) and we can apply Theorem 3.4.3 to solve the linear system (3.2.12) without restriction on the period T. Let $\mathbf{W}$ be the set defined by

$$\begin{aligned}\mathbf{W} := {}& \mathcal{C}^\theta_\sharp([0,T];\mathbf{L}^2(\Omega)) \times \left(\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{L}^2(\Omega)) \cap \mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega))\right)\\ &\times \mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o)) \times \mathcal{C}^\theta_\sharp([0,T];L^2(\Gamma_s)).\end{aligned}$$

The regularity space for the beam is denoted by

$$\mathcal{C}^\theta_{\text{beam}} := \mathcal{C}^\theta_\sharp([0,T];H^4(\Gamma_s)\cap H^2_0(\Gamma_s)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T];H^2_0(\Gamma_s)) \cap \mathcal{C}^{2+\theta}_\sharp([0,T];L^2(\Gamma_s)).$$

Theorem 3.2.6. For all $T > 0$ and $(\mathbf{f},\mathbf{w},\Theta,h) \in \mathbf{W}$, (3.2.8) admits a unique periodic solution

$$(\mathbf{u},p,\eta) \in \left(\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{L}^2(\Omega)) \cap \mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^2(\Omega))\right) \times \mathcal{C}^\theta_\sharp([0,T];H^1(\Omega)) \times \mathcal{C}^\theta_{\text{beam}}.$$

Moreover $(\mathbf{u}(0),\eta(0),\eta_t(0))$ is given by

$$\mathbf{u}(0) = \Pi\mathbf{v}(0) + \nabla N_s(\eta_t(0)) + \mathbf{w}(0) + L_{\Gamma_i,1}(\boldsymbol{\omega}_1)(0) \ \text{ and } \begin{pmatrix}\Pi\mathbf{v}(0)\\ \eta(0)\\ \eta_t(0)\end{pmatrix} = P_{\mathcal{A}}\mathbf{f},$$

where $P_{\mathcal{A}}$ is defined in Lemma 3.4.2. Finally, the following estimate holds

$$\begin{aligned}&\|\mathbf{u}\|_{\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{L}^2(\Omega))\cap\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^2(\Omega))} + \|p\|_{\mathcal{C}^\theta_\sharp([0,T];H^1(\Omega))} + \|\eta\|_{\mathcal{C}^\theta_{\text{beam}}}\\ &\le C_L\Big[\|\boldsymbol{\omega}_1\|_{\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i))\cap\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o))}\\ &\qquad + \|(\mathbf{f},\mathbf{w},\Theta,h)\|_{\mathbf{W}}\Big].\end{aligned}\tag{3.2.16}$$

3.3 Nonlinear system

In this section we prove the existence of classical solutions for the nonlinear system (3.1.3) using a fixed point argument. Without additional source terms in the model, here given through the inflow and outflow boundary conditions, the solution obtained with the fixed point procedure is the null solution. Hence, in what follows, the pair $(\boldsymbol{\omega}_1,\omega_2)$ is assumed to be non trivial, eventually small enough, and represents the 'impulse' of the system. The period T of $(\boldsymbol{\omega}_1,\omega_2)$ determines the period of the whole system.

Let $T \in \mathbb{R}^+ \setminus \{\frac{2k\pi}{b_j} \mid k \in \mathbb{Z},\ 0 \leq j \leq N_{\mathcal{A}}\}$ be a fixed time and

$$(\boldsymbol{\omega}_1, \omega_2) \in \Big(\mathcal{C}_\sharp^\theta([0,T]; \mathbf{H}_0^{3/2}(\Gamma_i)) \cap \mathcal{C}_\sharp^{1+\theta}([0,T]; \mathbf{H}^{-1/2}(\Gamma_i))\Big) \times \mathcal{C}_\sharp^\theta([0,T]; H^{1/2}(\Gamma_o)).$$

Consider the Banach space $\mathcal{X}$ defined by

$$\mathcal{X} = \Big(\mathcal{C}_\sharp^{1+\theta}([0,T]; \mathbf{L}^2(\Omega)) \cap \mathcal{C}_\sharp^\theta([0,T]; \mathbf{H}^2(\Omega))\Big) \times \mathcal{C}_\sharp^\theta([0,T]; H^1(\Omega)) \times \mathcal{C}_{\mathrm{beam}}^\theta,$$

and

$$\mathcal{B}(R,\mu) = \{(\mathbf{u}, p, \eta) \in \mathcal{X} \mid \|(\mathbf{u}, p, \eta)\|_{\mathcal{X}} \leq R,\ \left\|(1+\eta)^{-1}\right\|_{\mathcal{C}([0,T]\times\Gamma_s)} \leq \mu\},$$

with $R > 0$ and $\mu > 0$.

Theorem 3.3.1. Let $R > 0$, $\mu > 0$ and $(\mathbf{u}, p, \eta) \in \mathcal{B}(R,\mu)$. There exists a polynomial $Q \in \mathbb{R}^+[X]$ satisfying $Q(0) = 0$ and a constant $C(\mu) > 0$ such that the following estimates hold

$$|| \underbrace{(\mathbf{G}(\mathbf{u}, p, \eta), \mathbf{w}(\mathbf{u}, \eta), (1/2)|\mathbf{u}|^2, \Psi(\mathbf{u}, \eta))}_{=:\mathbf{F}(\mathbf{u},p,\eta)} ||_{\mathbf{W}} \leq C(\mu) Q(R) \|(\mathbf{u}, p, \eta)\|_{\mathcal{X}},$$

and for $(\mathbf{u}_i, p_i, \eta_i) \in \mathcal{B}(R,\mu)$ $(i = 1, 2)$

$$\|\mathbf{F}(\mathbf{u}_1, p_1, \eta_1) - \mathbf{F}(\mathbf{u}_2, p_2, \eta_2)\|_{\mathbf{W}} \leq C(\mu) Q(R) \|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{\mathcal{X}} \tag{3.3.1}$$

Proof. The nonlinear terms $(\mathbf{G}(\mathbf{u}, p, \eta), \mathbf{w}(\mathbf{u}, \eta), (1/2)|\mathbf{u}|^2, \Psi(\mathbf{u}, p))$ were already estimated in Theorem 2.4.1 with explicit time dependency for Sobolev regularity in time. Here the time dependency is straightforward as all the functions involved in the estimates are Hölder continuous and T is fixed. For example:

$$\|\eta \mathbf{u}_t\|_{\mathcal{C}^\theta([0,T];\mathbf{L}^2(\Omega))} = \|\eta \mathbf{u}_t\|_{\mathcal{C}([0,T];\mathbf{L}^2(\Omega))} + \sup_{t_1 \neq t_2} \frac{\|\eta(t_1)\mathbf{u}_t(t_1) - \eta(t_2)\mathbf{u}_t(t_2)\|_{\mathbf{L}^2(\Omega)}}{|t_1 - t_2|^\theta},$$

and the following estimates hold

$$\|\eta \mathbf{u}_t\|_{\mathcal{C}([0,T];\mathbf{L}^2(\Omega))} \leq \|\eta\|_{\mathcal{C}([0,T];L^\infty(\Gamma_s))} \|\mathbf{u}_t\|_{\mathcal{C}([0,T];\mathbf{L}^2(\Omega))}$$

$$\begin{aligned}
&\sup_{t_1 \neq t_2} \frac{\|\eta(t_1)\mathbf{u}_t(t_1) - \eta(t_2)\mathbf{u}_t(t_2)\|_{\mathbf{L}^2(\Omega)}}{|t_1 - t_2|^\theta} \\
&\leq \sup_{t_1 \neq t_2} \frac{\|\eta(t_1) - \eta(t_2)\|_{L^\infty(\Gamma_s)}}{|t_1 - t_2|^\theta} \|\mathbf{u}_t\|_{\mathcal{C}([0,T];\mathbf{L}^2(\Omega))} + \sup_{t_1 \neq t_2} \frac{\|\mathbf{u}_t(t_1) - \mathbf{u}_t(t_2)\|_{\mathbf{L}^2(\Omega)}}{|t_1 - t_2|^\theta} \|\eta\|_{\mathcal{C}([0,T];L^\infty(\Gamma_s))} \\
&\leq \|\eta\|_{\mathcal{C}^\theta([0,T];H^4(\Gamma_s))} \|\mathbf{u}_t\|_{\mathcal{C}([0,T];\mathbf{L}^2(\Omega))} + \|\mathbf{u}_t\|_{\mathcal{C}^\theta([0,T];\mathbf{L}^2(\Omega))} \|\eta\|_{\mathcal{C}([0,T];L^\infty(\Gamma_s))} \\
&\leq 2 \|\eta\|_{\mathcal{C}^\theta([0,T];H^4(\Gamma_s))} \|\mathbf{u}_t\|_{\mathcal{C}^\theta([0,T];\mathbf{L}^2(\Omega))}.
\end{aligned}$$

The other 'ball' estimates and the Lipschitz estimates (3.3.1) are obtained through the same techniques using the following Sobolev embeddings

$$\begin{aligned}&\|\eta\|_{\mathcal{C}^\theta([0,T];L^\infty(\Gamma_s))} + \|\eta_x\|_{\mathcal{C}^\theta([0,T];L^\infty(\Gamma_s))} + \|\eta_{xx}\|_{\mathcal{C}^\theta([0,T];L^\infty(\Gamma_s))} + \|\eta_{xxx}\|_{\mathcal{C}^\theta([0,T];L^\infty(\Gamma_s))}\\ &+ \|\eta_t\|_{\mathcal{C}^\theta([0,T];L^\infty(\Gamma_s))} + \|\eta_{tx}\|_{\mathcal{C}^\theta([0,T];L^\infty(\Gamma_s))} \leq C\, \|\eta\|_{\mathcal{C}^\theta([0,T];H^4(\Gamma_s))\cap\mathcal{C}^{1+\theta}([0,T];H^2(\Gamma_s))}\,.\end{aligned}$$

Finally remarks that all the nonlinear terms are at least quadratic and thus are bounded by $\|(\mathbf{u},p,\eta)\|^\alpha_{\mathcal{X}}$ for $\alpha \geq 2$. Writing $\|(\mathbf{u},p,\eta)\|^\alpha_{\mathcal{X}} \leq R^{\alpha-1}\,\|(\mathbf{u},p,\eta)\|_{\mathcal{X}}$, with $\alpha - 1 \geq 1$, concludes the proof. □

For $R > 0$ and $\mu > 0$ introduce the map

$$\mathcal{F}: \begin{cases}\mathcal{B}(R,\mu) & \longrightarrow \mathcal{X},\\ (\mathbf{u},p,\eta) & \longmapsto (\mathbf{u}^*,p^*,\eta^*),\end{cases} \tag{3.3.2}$$

where $(\mathbf{u}^*,p^*,\eta^*)$ is the solution to (3.2.8) with right-hand side

$$(\mathbf{f},\mathbf{w},\Theta,h) = (\mathbf{G}(\mathbf{u},p,\eta),\mathbf{w}(\mathbf{u},\eta),(1/2)|\mathbf{u}|^2,\Psi(\mathbf{u},\eta)).$$

Theorem 3.3.2. There exists $R^* > 0$ and $\mu^* > 0$ such that for all

$$(\boldsymbol{\omega}_1,\omega_2) \in \Big(\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))\Big) \times \mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o)),$$

satisfying

$$\|\boldsymbol{\omega}_1\|_{\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i))\cap\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o))} \leq \frac{R^*}{2C_L},$$

where C_L is the constant involved in (3.2.16), system (3.1.3) admits a unique solution $(\mathbf{u},p,\eta)$ in the ball $\mathcal{B}(R^*,\mu^*)$.

Proof. Let $R_1 > 0$ and $\mu^* > 1$. In order to ensure that the map $\mathcal{F}$ is well defined from $\mathcal{B}(R^*,\mu^*)$ into itself (with R^* to be defined) we control the estimate on $\|(1+\eta)^{-1}\|_{\mathcal{C}([0,T]\times\Gamma_s)}$ with the parameter R_1. Precisely, for all $(\mathbf{u},p,\eta) \in \mathcal{B}(R_1,\mu^*)$, the following estimate holds

$$\|\eta\|_{\mathcal{C}([0,T]\times\Gamma_s,\Gamma_s)} \leq C_\infty R_1,$$

with $C_\infty > 0$ a positive constant. Then we choose $R_2 < \frac{\mu^*-1}{C_\infty\mu^*}$ and for all $(\mathbf{u},p,\eta) \in \mathcal{B}(R_2,\mu^*)$ the following estimate holds

$$\left\|(1+\eta)^{-1}\right\|_{\mathcal{C}([0,T]\times\Gamma_s,\Gamma_s)} \leq \frac{1}{1-C_\infty R_2} < \mu^*.$$

The linear estimate 3.2.16 implies that, for all $(\mathbf{u},p,\eta) \in \mathcal{B}(R_2,\mu^*)$,

$$\begin{aligned}&\|\mathcal{F}(\mathbf{u},p,\eta)\|_{\mathcal{X}} \leq C_L(\|\boldsymbol{\omega}_1\|_{\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i))\cap\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o))}\\ &+ C(\mu^*)Q(R_2)\,\|(\mathbf{u},p,\eta)\|_{\mathcal{X}}).\end{aligned}$$

We choose $0 < R^* < R_2$ such that $C(\mu^*)Q(R^*) < \min(\frac{1}{2C_L}, \frac{1}{2})$. Finally choose $(\boldsymbol{\omega}_1, \omega_2)$ such that

$$\|\boldsymbol{\omega}_1\|_{\mathcal{C}^\theta_\sharp([0,T];\mathbf{H}^{3/2}_0(\Gamma_i))\cap\mathcal{C}^{1+\theta}_\sharp([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^\theta_\sharp([0,T];H^{1/2}(\Gamma_o))} \leq \frac{R^*}{2C_L}.$$

At this point we proved that $\mathcal{F}$ is well defined from $\mathcal{B}(R^*, \mu^*)$ into itself. Moreover, using (3.3.1), the Lipschitz estimate

$$\begin{aligned}\|\mathcal{F}(\mathbf{u}_1, p_1, \eta_1) - \mathcal{F}(\mathbf{u}_2, p_2, \eta_2)\|_{\mathcal{X}} &\leq C_L C(\mu^*)Q(R^*)\, \|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{\mathcal{X}} \\ &\leq \frac{1}{2}\, \|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{\mathcal{X}}\,.\end{aligned}$$

for all $(\mathbf{u}_i, p_i, \eta_i) \in \mathcal{B}(R^*, \mu)$ $(i = 1, 2)$ shows that $\mathcal{F}$ is a contraction from $\mathcal{B}(R^*, \mu^*)$ into itself. The Banach fixed point theorem then ensures the existence of a solution to (3.1.3). □

Remark 3.3.1. Notice that all the previous work can be done similarly with data $(\boldsymbol{\omega}_1, \omega_2)$ in $\left(L^2(0,T;\mathbf{H}^{3/2}_0(\Gamma_i)) \cap H^1_\sharp(0,T;\mathbf{H}^{-1/2}(\Gamma_i))\right) \times L^2(0,T;H^{1/2}(\Gamma_o))$. Indeed the existence of a solution for the linear system is similar and the nonlinear estimates are provided in Theorem 2.4.1. We obtained a solution

$$(\mathbf{u}, p, \eta) \in \mathbf{H}^{2,1}_\sharp(Q_T) \times L^2(0,T;H^1(\Omega)) \times H^{4,2}_\sharp(\Sigma^s_T).$$

This proof of existence also applies to other boundary conditions. For instance, as soon as the Stokes problem admits a solution in $\mathbf{H}^2(\Omega)$ (e.g. for pressure boundary conditions on the inflow and the outflow, Dirichlet boundary condition, periodic boundary conditions...) the results are valid.

3.4 Appendix: Abstract results on periodic evolution equations

Let H be a Hilbert space (with norm $\|\cdot\|$) and A be the infinitesimal generator of an analytic semigroup $S(t)$ on H with domain $\mathcal{D}(A)$. In this section we are interested in the existence of a T-periodic solution to the following abstract evolution equation

$$y'(t) = Ay(t) + f(t)\,,\ \text{for } t \in \mathbb{R}, \tag{3.4.1}$$

where $f : \mathbb{R} \to H$ is a T-periodic source term with a regularity to be specified. A T-periodic function y is solution to (3.4.1) if and only if its restriction to $[0,T]$ is solution to

$$\begin{cases} y'(t) = Ay(t) + f(t)\,,\ \text{for all } t \in [0,T], \\ y(0) = y(T). \end{cases} \tag{3.4.2}$$

In this section, two frameworks are considered to study (3.4.2). The Hilbert case, when $f \in L^2(0,T;H)$, and the continuous case when $f \in \mathcal{C}([0,T];H)$ (or f is Hölder continuous). The Hilbert case provides powerful tools to study (3.4.2) through the existence of isomorphism theorems [10, Theorem 3.1, part II, section 1.3]. This framework is used to prove the existence of a unique solution to (3.4.2) under additional hypothesis on the operator A. The previous strategy is developed in Section 3.4.1. When f is continuous or Hölder continuous, we use the continuous theory for evolution equations to improve the regularity of this solution. In both case we are interested in the existence of strict solutions. For $y^0 \in H$ and $f \in L^2(0,T;H)$ consider the evolution equation

$$\begin{cases} y'(t) = Ay(t) + f(t)\,, \text{ for all } t \in [0,T], \\ y(0) = y^0. \end{cases} \tag{3.4.3}$$

Definition 3.4.1.

(i) y is a strict solution of (3.4.3) in $L^2(0,T;H)$ if y belongs to $L^2(0,T;\mathcal{D}(A)) \cap H^1(0,T;H)$, $y'(t) = Ay(t) + f(t)$ for a.e. $t \in [0,T]$, and $y(0) = y^0$.

(ii) y is a strict solution of (3.4.3) in $\mathcal{C}([0,T];H)$ if y belongs to $\mathcal{C}([0,T];\mathcal{D}(A)) \cap \mathcal{C}^1([0,T];H)$, $y'(t) = Ay(t) + f(t)$ for all $t \in [0,T]$, and $y(0) = y^0$.

(iii) y is a classical solution of (3.4.3) in $\mathcal{C}([0,T];H)$ if y belongs to $\mathcal{C}((0,T];\mathcal{D}(A)) \cap \mathcal{C}^1((0,T];H) \cap \mathcal{C}([0,T];H)$, $y'(t) = Ay(t) + f(t)$ for all $t \in [0,T]$, and $y(0) = y^0$.

(iv) The function

$$y(t) = S(t)y^0 + \int_0^t S(t-s)f(s)ds,$$

is called the mild solution of problem (3.4.3) if y belongs to $\mathcal{C}([0,T];H)$.

In what follows we assume that the pair (A,T) satisfies the assumption:

(3.4.4)

The resolvent of A is compact, $0 \notin \sigma_p(A)$ and $T \in \mathbb{R}^+ \setminus \{\dfrac{2k\pi}{b_j} \mid k \in \mathbb{Z},\ 0 \le j \le N_A\}$

where $\{ib_j\}_{0 \le j \le N_A}$ denote the non zero eigenvalues of A on the imaginary axis $i\mathbb{R}$ with $N_A \in \mathbb{N}$ and $b_j \in \mathbb{R}^*$ with $0 \le j \le N_A$.

Remark that the assumptions A generates an analytic semigroup and has a compact resolvent directly imply that N_A is a finite number.

3.4.1 Hilbert case

In this section we obtain a simple criteria to ensure that the problem (3.4.2) admits a unique strict solution in $L^2(0,T;H)$.

Lemma 3.4.1. The evolution equation (3.4.2) admits a strict solution in $L^2(0,T;H)$ if and only if the equation

$$(I - S(T))z = \int_0^T S(T-s)f(s)ds. \tag{3.4.5}$$

admits at least one solution $z \in [\mathcal{D}(A), H]_{1/2}$.

Proof. Suppose that (3.4.2) admits a strict solution $y \in L^2(0,T;\mathcal{D}(A)) \cap H^1(0,T;H)$. We recall, see [38], that $L^2(0,T;\mathcal{D}(A)) \cap H^1(0,T;H) \subset \mathcal{C}([0,T];[\mathcal{D}(A),H]_{1/2})$. As this solution coincides with the mild solution given by the Duhamel formula we have

$$y(0) = y(T) = S(T)y(0) + \int_0^T S(T-s)f(s)ds,$$

and thus $z = y(0)$ satisfies (3.4.5). Reciprocally if $z \in [\mathcal{D}(A), H]_{1/2}$ satisfies the equation (3.4.5) then consider the evolution equation

$$\begin{cases} y'(t) = Ay(t) + f(t)\, , \text{ for all } t \in [0,T], \\ y(0) = z. \end{cases} \tag{3.4.6}$$

The isomorphism theorem [10, Theorem 3.1, part II, section 1.3] shows that (3.4.6) admits a unique solution $y \in L^2(0,T;\mathcal{D}(A)) \cap H^1(0,T;H)$. Finally this solution satisfies (3.4.2) by choice of z. □

Introduce the function v defined by

$$v(t) = \int_0^t S(t-s)f(s)ds,$$

and remark that $v(t) \in [\mathcal{D}(A), H]_{1/2}$ for all $t \in [0,T]$.

Lemma 3.4.2. Suppose that the pair (A,T) satisfies the assumption (3.4.4). Then the equation (3.4.5) admits a unique solution $z \in [\mathcal{D}(A), H]_{1/2}$. Moreover the operator P_A defined by

$$P_A f = (I - S(T))^{-1} \int_0^T S(T-s)f(s)ds,$$

is a bounded linear operator from $L^2(0,T;H)$ into $[\mathcal{D}(A), H]_{1/2}$.

Proof. First remark that as the semigroup $S(t)$ is analytic we have $S(T)z \in \mathcal{D}(A^n)$ for all $n \geq 0$ and $z \in H$. Hence a solution z to (3.4.5) has the same regularity to $v(T)$ i.e. is in $[\mathcal{D}(A), H]_{1/2}$.

The assumption that A has a compact resolvent implies (see [52, Theorem 3.3] and recall that $S(t)$ is analytic and thus differentiable, which implies the continuity for the uniform operator topology for $t > 0$) that $S(t)$ is a compact semigroup. Hence $\sigma(S(T)) = \sigma_p(S(T))$ and the spectral mapping theorem $e^{T\sigma_p(A)} = \sigma_p(S(T))$, coupled with the assumption (3.4.4), shows that $1 \in \rho(S(T))$. Thus $(I - S(T))z = w$ for $w \in [\mathcal{D}(A), H]_{1/2} \subset H$ can be rewritten $z = (I - S(T))^{-1}w \in H$ and this z belongs to $[\mathcal{D}(A), H]_{1/2}$. We have proved that the operator $(I - S(T))$ is a bijection from $[\mathcal{D}(A), H]_{1/2}$ into itself. By definition $S(T) \in \mathcal{L}(H)$. Moreover, using the graph norm on $\mathcal{D}(A)$ and a classical estimate for analytic semigroups, we have for all $u \in \mathcal{D}(A)$

$$\|S(T)u\|_{\mathcal{D}(A)} = \|S(T)u\|_H + \|AS(T)u\|_H \leq \|S(T)\|_{\mathcal{L}(H)} \|u\|_H + \frac{C}{T} \|u\|_H \leq C \|u\|_{\mathcal{D}(A)} .$$

Hence $(I - S(T)) \in \mathcal{L}(\mathcal{D}(A))$ and by interpolation $(I - S(T)) \in \mathcal{L}([\mathcal{D}(A), H]_{1/2})$. Finally the bounded inverse theorem implies that $(I - S(T))^{-1}$ is a bounded linear operator on $[\mathcal{D}(A), H]_{1/2}$. From the continuous embedding $L^2(0, T; \mathcal{D}(A)) \cap H^1(0, T; H) \subset \mathcal{C}^0([0, T]; [\mathcal{D}(A), H]_{1/2})$ we obtain that

$$\|v(T)\|_{[\mathcal{D}(A),H]_{1/2}} \leq C(\|v\|_{L^2(0,T;\mathcal{D}(A))} + \|v\|_{H^1(0,T;H)}) \leq C \|f\|_{L^2(0,T;H)} ,$$

and

$$\|P_A f\|_{[\mathcal{D}(A),H]_{1/2}} \leq C \left\|(I - S(T))^{-1}\right\|_{\mathcal{L}([\mathcal{D}(A),H]_{1/2})} \|f\|_{L^2(0,T;H)} .$$

□

Hence we have proved the following theorem.

Theorem 3.4.1. Suppose that the pair (A, T) satisfies the assumption (3.4.4). Then the periodic evolution equation (3.4.2) admits a unique strict solution $y \in L^2(0, T; \mathcal{D}(A)) \cap H^1_\sharp(0, T; H)$ in $L^2(0, T; H)$. The following estimate holds

$$\|y\|_{L^2(0,T;\mathcal{D}(A)) \cap H^1_\sharp(0,T;H)} \leq C \|f\|_{L^2(0,T;H)} .$$

Proof. It remains to prove the estimate. Using [10, Theorem 3.1, part II, section 1.3] and Lemma 3.4.2 we obtain

$$\begin{aligned}
\|y\|_{L^2(0,T;\mathcal{D}(A)) \cap H^1_\sharp(0,T;H)} &\leq C(\left\|y^0\right\|_{[\mathcal{D}(A),H]_{1/2}} + \|f\|_{L^2(0,T;H)}) \\
&\leq C(\|P_A f\|_{[\mathcal{D}(A),H]_{1/2}} + \|f\|_{L^2(0,T;H)}) \\
&\leq C \|f\|_{L^2(0,T;H)} .
\end{aligned}$$

□

Using the regularization properties of analytic semigroup for $t > 0$, that is $S(t)z \in \mathcal{D}(A^n)$ for all $n \geq 1$ and $z \in H$, we can prove that the regularity of the solution solely depends on the source term f. Hence the previous result can be improved when f is more regular. We introduce the space $\mathcal{H}^r = [\mathcal{D}(A^{n+1}), \mathcal{D}(A^n)]_{1-\alpha}$ with $r = n + \alpha$, $n \geq 0$ an integer and $0 \leq \alpha \leq 1$ a real number.

Lemma 3.4.3. Let f be in $L^2(0,T;\mathcal{H}^{r-1})$ with $r>1$ and suppose that the pair (A,T) satisfies the assumption (3.4.4). Then the unique strict solution y in $L^2(0,T;H)$ belongs to $y\in L^2(0,T;\mathcal{H}^r)\cap H^1_\sharp(0,T;\mathcal{H}^{r-1})$ and $y(0)\in[\mathcal{H}^r,\mathcal{H}^{r-1}]_{1/2}$.

Proof. We split (3.4.2) in two parts

$$\begin{cases} y_1'(t) = Ay_1(t) + f(t)\,, \text{ for all } t\in[0,T], \\ y_1(0) = 0, \end{cases}$$

and

$$\begin{cases} y_2'(t) = Ay_2(t)\,, \text{ for all } t\in[0,T], \\ y_2(0) = z. \end{cases}$$

Using the analyticity of S we have $y_2(T) = S(T)z \in \mathcal{D}(A^n)$ for all $n\geq 1$. On the other hand [10, Theorem 2.2, part II, section 3.2.1] (and the remark following the theorem on the extension of the isomorphism theorem) implies that $y_1 \in L^2(0,T;H^r)\cap H^1(0,T;H^{r-1})\subset\mathcal{C}^0([0,T];[H^r,H^{r-1}]_{1/2})$. Hence $y(T)=y_1(T)+y_2(T)$ is in $[H^r,H^{r-1}]_{1/2}$. Then we use the periodic condition $y(T)=y(0)$ and again [10, Theorem 2.2, part II, section 3.2.1] to obtain $y\in L^2(0,T;\mathcal{H}^r)\cap H^1_\sharp(0,T;\mathcal{H}^{r-1})$. □

3.4.2 Continuous case

Let us recall the fundamental existence and regularity result (see [4, Theorem 1.2.1, Section II]):

Theorem 3.4.2. Suppose that $f\in\mathcal{C}^\rho([0,T];H)$ with $\rho\in(0,1)$ and $y^0\in H$. Then the Cauchy problem (3.4.3) possesses a unique classical solution y in $\mathcal{C}([0,T];H)$ and

$$y\in\mathcal{C}^\rho((0,T];\mathcal{D}(A))\cap\mathcal{C}^{\rho+1}((0,T];H),$$

with the estimate, for all $\varepsilon>0$,

$$\|y\|_{\mathcal{C}^\rho([\varepsilon,T];\mathcal{D}(A))\cap\mathcal{C}^{\rho+1}([\varepsilon,T];H)} \leq C(\|y(\varepsilon)\|_{\mathcal{D}(A)} + \|f\|_{\mathcal{C}^\rho([0,T];H)}).$$

If $y^0\in\mathcal{D}(A)$ then the solution is strict.

Proof. The estimate can be obtained following the steps of the proof in [4, Theorem 1.2.1, Section II], see in particular [4, Theorem 2.5.6, Section III]. □

We are now able to prove the existence of a strict periodic solution in $\mathcal{C}([0,T];H)$. Moreover, the previous Hölder regularity result and the periodicity show that the periodic solution possesses Hölder regularity up to $t=0$.

Theorem 3.4.3. Let $f \in \mathcal{C}^\rho([0,T];H)$ with $\rho \in (0,1)$ and suppose that the pair (A,T) satisfies the assumption (3.4.4). Then the periodic evolution equation (3.4.2) admits a unique strict solution y in $\mathcal{C}([0,T];H)$. Moreover

$$y \in \mathcal{C}^\rho([0,T];\mathcal{D}(A)) \cap \mathcal{C}^{\rho+1}([0,T];H),$$

and the following estimate holds

$$\|y\|_{\mathcal{C}^\rho([0,T];\mathcal{D}(A))\cap\mathcal{C}^{\rho+1}([0,T];H)} \leq C\, \|f\|_{\mathcal{C}^\rho([0,T];H)}\,. \tag{3.4.7}$$

Proof. We already know that there exists a strict solution in $L^2(0,T;H)$. Keeping the notations used in Lemma 3.4.3, we split $y = y_1 + y_2$. For y_2 we still have $y_2(T) \in \mathcal{D}(A)$. Theorem 3.4.2 implies that $y_1 \in \mathcal{C}^\rho((0,T];\mathcal{D}(A)) \cap \mathcal{C}^{\rho+1}((0,T];H)$, thus $y_1(T) \in \mathcal{D}(A)$. Then the periodic condition $y(0) = y(T)$ implies that $y(0) \in \mathcal{D}(A)$ and Theorem 3.4.2 ensures the existence of a strict solution in $\mathcal{C}([0,T];H)$. Finally, considering the T-periodic extension $\hat{y}$ of y on $[0,2T]$ the Hölder regularity result implies that $\hat{y} \in \mathcal{C}^\rho((0,2T];\mathcal{D}(A)) \cap \mathcal{C}^{\rho+1}((0,2T];H)$. Hence $\hat{y}$ is Hölder in a neighbourhood of T, which implies that $y \in \mathcal{C}^\rho([0,T];\mathcal{D}(A)) \cap \mathcal{C}^{\rho+1}([0,T];H)$. It remains to estimate y with respect to f. Let us fix $\varepsilon = \frac{T}{2}$. We have

$$y(\varepsilon) = S(\varepsilon)y^0 + \int_0^\varepsilon S(\varepsilon - s)f(s)ds.$$

The homogeneous part was already estimated in Lemma 3.4.2

$$\left\|S(\varepsilon)y^0\right\|_{\mathcal{D}(A)} \leq C\left\|y^0\right\|_H.$$

The integral part in Duhamel can be estimated as follows

$$\int_0^\varepsilon AS(\varepsilon-s)f(s)ds = \int_0^\varepsilon AS(\varepsilon-s)(f(s)-f(\varepsilon))ds + \int_0^\varepsilon AS(\varepsilon-s)f(\varepsilon)ds,$$

and

$$\left\|\int_0^\varepsilon AS(\varepsilon-s)f(\varepsilon)ds\right\|_H = \|(S(\varepsilon)-I)f(\varepsilon)\|_H \leq C\,\|f\|_{\mathcal{C}^\rho([0,T];H)}\,,$$

where we have used $\dfrac{d}{dt}S(t) = AS(t)$. Finally

$$\left\|\int_0^\varepsilon AS(\varepsilon-s)(f(s)-f(\varepsilon))ds\right\|_H \leq \int_0^\varepsilon \frac{C}{|\varepsilon-s|}|\varepsilon-s|^\rho\,\|f\|_{\mathcal{C}^\rho([0,T];H)}\,ds \leq C\,\|f\|_{\mathcal{C}^\rho([0,T];H)}\,,$$

and $\|y(\varepsilon)\|_{\mathcal{D}(A)} \leq C\,\|f\|_{\mathcal{C}^\rho([0,T];H)}$. The estimate in Theorem 3.4.2 implies that

$$\|\hat{y}\|_{\mathcal{C}^\rho([\varepsilon,2T];\mathcal{D}(A))\cap\mathcal{C}^{\rho+1}([\varepsilon,2T];H)} \leq C\left\|\hat{f}\right\|_{\mathcal{C}^\rho([0,2T];H)}\,,$$

where $\hat{f}$ is the T-periodic extension of f to $[0,2T]$. Then, taking the restriction to a period T, we obtain the estimate (3.4.7). $\square$

Chapter 4

Stabilization of a time-periodic fluid–structure system

4.1 Introduction

4.1.1 Setting of the problem

We study the stabilization of a fluid-structure system coupling the Navier–Stokes equations with an Euler–Bernoulli beam equation around a T-periodic solution with $T > 0$. For $L > 0$ consider the domain Ω in $\mathbb{R}^2$ defined by $\Omega = (0, L) \times (0, 1)$. The different components of the boundary $\partial\Omega$ are denoted by: $\Gamma_i = \{0\} \times (0, 1)$, $\Gamma_o = \{L\} \times (0, 1)$, $\Gamma_b = (0, L) \times \{0\}$, $\Gamma_s = (0, L) \times \{1\}$. The domain of the fluid at the time $t \geq 0$ is denoted by $\Omega_{\eta(t)}$ and depends on the displacement of the beam $\eta : \Gamma_s \times (0, +\infty) \to (-1, +\infty)$. More precisely

$$\begin{aligned}
\Omega_{\eta(t)} &= \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ 0 < y < 1 + \eta(x, 1, t)\},\\
\Gamma_{\eta(t)} &= \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ y = 1 + \eta(x, 1, t)\}.
\end{aligned}$$

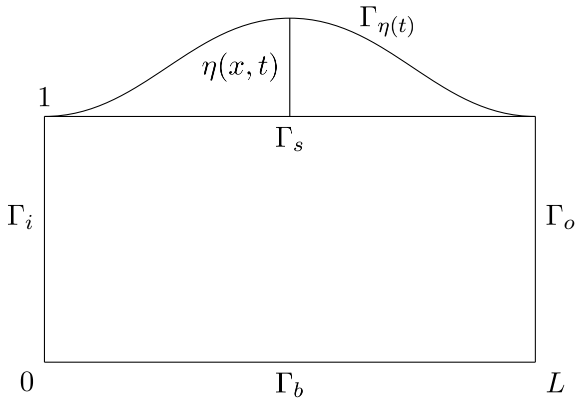

Figure 4.1: Fluid–structure system.

For space-time domain we use the notations

$$\Sigma_\infty^s = \Gamma_s \times (0,\infty),\ \Sigma_\infty^i = \Gamma_i \times (0,\infty),\ \Sigma_\infty^o = \Gamma_o \times (0,\infty),\ \Sigma_\infty^b = \Gamma_b \times (0,\infty),$$
$$\Sigma_\infty^\eta = \bigcup_{t\in(0,\infty)} \Gamma_{\eta(t)} \times \{t\},\ Q_\infty^\eta = \bigcup_{t\in(0,\infty)} \Omega_{\eta(t)} \times \{t\}.$$

The previous notations are also used with $+\infty$ replaced by a finite time. The fluid structure system is described by the following equations.

$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u},p) = 0,\ \operatorname{div}\mathbf{u} = 0 \text{ in } Q_\infty^\eta,\\
&\mathbf{u}(x,y,t) = \eta_t(x,1,t)\mathbf{e}_2 \text{ for } (x,y,t)\in\Sigma_\infty^\eta,\\
&\mathbf{u} = \boldsymbol{\omega}_1 + \mathbf{u}_c \text{ on } \Sigma_\infty^i,\\
&u_2 = 0 \text{ and } p = \omega_2 \text{ on } \Sigma_\infty^o,\\
&\mathbf{u} = 0 \text{ on } \Sigma_\infty^b,\ \mathbf{u}(0) = \mathbf{u}^0 \text{ in } \Omega_{\eta_1^0},\\
&\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = -J_{\eta(t)}\mathbf{e}_2\cdot\sigma(\mathbf{u},p)_{|\Gamma_{\eta(t)}}\mathbf{n}_{\eta(t)} \text{ on } \Sigma_\infty^s\\
&\eta = 0 \text{ and } \eta_x = 0 \text{ on } \{0,L\}\times(0,\infty),\\
&\eta(0) = \eta_1^0 \text{ and } \eta_t(0) = \eta_2^0 \text{ in } \Gamma_s,
\end{aligned} \tag{4.1.1}$$

where $\mathbf{u} = (u_1,u_2)$ is the velocity, p the pressure, η the displacement of the beam and

$$\sigma(\mathbf{u},p) = -pI + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T),$$
$$\mathbf{n}_{\eta(t)} = J_{\eta(t)}^{-1}\begin{pmatrix}-\eta_x(x,t)\\ 1\end{pmatrix},$$

with $J_{\eta(t)} = \sqrt{1+\eta_x^2}$. The pair $(\boldsymbol{\omega}_1,\omega_2)$ represents a T-periodic forcing term for the system. The constants $\beta \geq 0$, $\gamma > 0$, $\alpha > 0$ are parameters relative to the structure and $\nu > 0$ is the constant viscosity of the fluid. As in Chapter 2 we use the following abuse of notation for a function f defined on the flat domain Γ_s or on $(0, L)$: $f(x) = f(x, 1) = f(x, y)$ for $(x, y) \in (0, L) \times \mathbb{R}$. Let $(\mathbf{u}_\pi, p_\pi, \eta_\pi)$ be a time periodic solution to (4.1.1). Our purpose here is to stabilize (4.1.1) around this periodic solution through the Dirichlet boundary control $\mathbf{u}_c$. The analysis presented here is restricted to the linear system associated to the control problem. The stabilization of the nonlinear system using the ideas of this chapter is ongoing.

To our knowledge, the stabilization of this 2D fluid–structure model around a time-dependent solution is new. In [56], the system (4.1.1) with homogeneous Dirichlet boundary conditions on the inflow and the outflow is exponentially stabilized, locally around the zero solution. The feedback control corresponds to a force term in the beam equation. For a similar system with periodic boundary conditions and a structure driven by a damped wave equation, an exponential stabilization result around the zero state is proved in [44] with a Dirichlet control acting on the lower part of the domain. The stabilizability around a non zero stationary solution, for a fluid–structure system evolving in two dimensional polygonal domain with mixed boundary conditions, is established in [23]. As in [56], the feedback law corresponds to a force term in the beam equation. In [56] it is shown that the stabilization follows from a unique continuation property proved in [50]. In [23], due to the additional terms coming from the linearization around a non zero solution, the corresponding unique continuation problem leads to an open question in the field. Therefore, this property is an assumption in [23], which can be verified numerically.

Let us mention some references on the stabilization of abstract evolution equations. In [40, 42], the author develops an infinite dimensional Floquet theory for time dependent parabolic equations. These results are then used in [41] to study the stability of fully nonlinear parabolic equations. In these papers the control operator is bounded. The feedback stabilization of periodic parabolic equations with unbounded control is studied in [8]. We also refer to [20] for a general presentation of periodic evolution equations.

4.1.2 Plan of the chapter

The stabilization of (4.1.1) around a periodic solution presents several challenges. For each $t \geq 0$, the perturbed solution $(\mathbf{u}, p, \eta)$ evolves in $\Omega_{\eta(t)}$ whereas the periodic solution is written in $\Omega_{\eta_\pi(t)}$. To compare these two solutions we perform a change of variables to write the perturbed solution in the time-periodic domain $\Omega_{\eta_\pi(t)}$. We then linearize the difference of the two solutions around $(\mathbf{0}, 0, 0)$. We obtain a perturbed linear system involving the periodic solution.

The domain of the linear system still depends on the time. When the domain is fixed with a change of variables, the linear system is strongly perturbed. To study the spectrum of the underlying operator we require assumptions on the periodic solution. Precisely we suppose that the beam displacement of the periodic solution stays in a small 'cylinder' i.e. $\eta_\pi(t) - \eta_\pi(s)$ is small for all t, s. We choose a second change of variables mapping $\Omega_{\eta_\pi(t)}$ into $\Omega_{\eta_\pi(0)}$. As in Chapter 2, the coefficient $\eta_\pi(t) - \eta_\pi(0)$ appears in the additional linear terms. Using the smallness assumption on $\eta_\pi(t) - \eta_\pi(s)$, these terms can be initially removed, and then re-introduced as perturbation terms.

This first step is done in Section 4.2. The linear system (4.2.9) that we obtain depends on time. In Section 4.3 we introduce the tools required for our analysis of time-dependent parabolic equations. One of the key point when studying a parabolic system $y'(t) = A(t)y(t) + f(t)$ is the existence of a parabolic evolution operator associated to $A(t)$. We introduce the theory developed in [4] to construct an evolution operator when the domain of $A(t)$ may depend on t but a suitable interpolation space is independent of t. We then adapt the stabilization results presented in [42] to this framework when the graph norm of $\mathcal{D}(A(t))$ is equivalent to the norm of a fixed Banach space D.

The Section 4.4 presents preliminary results on a perturbed Oseen system which are used in Section 4.5 to obtain a matrix formulation of the linear system with an operator $\mathcal{A}(t)$. The existence of the parabolic evolution operator for $\mathcal{A}(t)$ is proved in Section 4.6. Finally we study the stabilization of the linear system through a Dirichlet boundary control in Section 4.7. The matrix formulation of the problem involves the control and its derivative. We consider an extended system to solve this issue and we prove that the system satisfies the Hautus criteria introduced in Section 4.3.

Throughout this chapter, the periodic solution $(\mathbf{u}_\pi, p_\pi, \eta_\pi)$ that we consider is more regular, in time, that the one constructed in Chapter 3. This is due to technical conditions in [4] and to the non-homogeneous divergence condition of the system after change of variables. However, and contrary to the solution obtained in Chapter 3, we do not require any smallness assumptions on this periodic solution, expect for the 'cylinder' condition on η_π.

4.2 System in the reference configuration

4.2.1 Function spaces

For η^0 belonging to $H^3(\Gamma_s) \cap H_0^2(\Gamma_s)$ and satisfying $1 + \eta^0(x) > 0$ for all $x \in (0, L)$ set

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ 0 < y < 1 + \eta^0(x)\},$$
$$\Gamma_{s,0} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L),\ y = 1 + \eta^0(x)\},$$

and $\Gamma_d = \Gamma_{s,0} \cup \Gamma_i \cup \Gamma_b$. To deal with the mixed boundary conditions, let us introduce the spaces

$$\mathbf{V}^0_{n,\Gamma_d}(\Omega_0) = \{\mathbf{v} \in \mathbf{L}^2(\Omega_0) \mid \text{div } \mathbf{v} = 0 \text{ in } \Omega_0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_d\},$$

and the orthogonal decomposition of $\mathbf{L}^2(\Omega_0) = L^2(\Omega_0, \mathbb{R}^2)$

$$\mathbf{L}^2(\Omega_0) = \mathbf{V}^0_{n,\Gamma_d}(\Omega_0) \oplus \text{grad } H^1_{\Gamma_o}(\Omega_0),$$

where $H^1_{\Gamma_o}(\Omega_0) = \{u \in H^1(\Omega_0) \mid u = 0 \text{ on } \Gamma_o\}$. Let $\Pi : \mathbf{L}^2(\Omega_0) \to \mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$ be the so-called Leray projector associated with this decomposition. If $\mathbf{u}$ belongs to $\mathbf{L}^2(\Omega_0)$ then $\Pi\mathbf{u} = \mathbf{u} - \nabla p_{\mathbf{u}} - \nabla q_{\mathbf{u}}$ where $p_{\mathbf{u}}$ and $q_{\mathbf{u}}$ are solutions to the following elliptic equations

$$\begin{aligned} &p_{\mathbf{u}} \in H^1_0(\Omega_0), \;\; \Delta p_{\mathbf{u}} = \text{div } \mathbf{u} \in H^{-1}(\Omega_0), \\ &q_{\mathbf{u}} \in H^1_{\Gamma_o}(\Omega_0), \;\; \Delta q_{\mathbf{u}} = 0, \;\; \frac{\partial q_{\mathbf{u}}}{\partial \mathbf{n}} = (\mathbf{u} - \nabla p_{\mathbf{u}}) \cdot \mathbf{n} \text{ on } \Gamma_d, \;\; q_{\mathbf{u}} = 0 \text{ on } \Gamma_o. \end{aligned} \tag{4.2.1}$$

Throughout this chapter the functions with vector values are written with a bold typography. For example $\mathbf{H}^2(\Omega_0) = H^2(\Omega_0, \mathbb{R}^2)$. As in Chapter 2 and using the notations in [38, Theorem 11.7], we introduce the space $H^{3/2}_{00}(\Gamma_s) = [H^1_0(\Gamma_s), H^2_0(\Gamma_s)]_{1/2}$. This space is a strict subspace of $H^{3/2}_0(\Gamma_s) = H^{3/2}(\Gamma_s) \cap H^1_0(\Gamma_s)$. Odd and even symmetries preserve the H^k-regularity for functions in $H^k_0(\Gamma_s)$ with $k = 1, 2$, thus, by interpolation, the $H^{3/2}$-regularity is also preserved for functions in $H^{3/2}_{00}(\Gamma_s)$. This property is used in Chapter 2 to handle the pressure boundary condition.

For the boundary condition on the inflow, we use the results developed in [47] for elliptic equations in a dihedron. In our case, the angle between Γ_i and $\Gamma_{s,0}$ is equal to $\frac{\pi}{2}$. If $\boldsymbol{\omega}$ (resp. $\mathbf{g}$) denotes the boundary condition on Γ_i (resp. $\Gamma_{s,0}$), the Laplace and Stokes equations possess solutions with H^2-regularity near $C_{0,1} = (0, 1)$ provided that the data are regular enough and that the compatibility conditions $\boldsymbol{\omega}(C_{0,1}) = \mathbf{g}(C_{0,1})$ is satisfied. To ensure these conditions, the non-homogeneous boundary condition on Γ_i is chosen in $H^{3/2}_0(\Gamma_i)$. Using the previous remarks, the following lemma is obtained as Lemma 2.3.1.

Lemma 4.2.1. Let $\mathbf{u} \in \mathbf{H}^2(\Omega_0)$ be such that

- div $\mathbf{u} = w$ in Ω_0 with $w \in H^1_{\Gamma_o}(\Omega_0)$,
- $\mathbf{u} = g\,\mathbf{e}_2$ on $\Gamma_{s,0}$ with $g \in H^{3/2}_{00}(\Gamma_{s,0})$ and $\mathbf{u} = \boldsymbol{\omega}$ on Γ_i with $\boldsymbol{\omega} \in \mathbf{H}^{3/2}_0(\Gamma_i)$,
- $\mathbf{u} = 0$ on Γ_b and $u_2 = 0$ on Γ_o.

Then $\Pi\mathbf{u}$ belongs to $\mathbf{H}^2(\Omega_0)$.

Consider the Stokes system

$$\begin{aligned} &-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \;\; \text{div } \mathbf{u} = 0 \;\text{ in } \Omega_0, \\ &\mathbf{u} = 0 \;\text{ on } \Gamma_d, \;\; u_2 = 0 \;\text{ and } p = 0 \;\text{ on } \Gamma_o. \end{aligned} \tag{4.2.2}$$

The energy space associated with (4.2.2) is

$$V = \{\mathbf{u} \in \mathbf{H}^1(\Omega_0) \mid \text{div } \mathbf{u} = 0 \text{ in } \Omega_0, \, \mathbf{u} = 0 \text{ on } \Gamma_d, \, u_2 = 0 \text{ on } \Gamma_o\}. \tag{4.2.3}$$

The regularity result for (4.2.2) is similar to Theorem 2.5.4, which enables us to define the Stokes operator $(A_s, \mathcal{D}(A_s))$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega_0)$ by

$$\mathcal{D}(A_s) = \mathbf{H}^2(\Omega_0) \cap V \text{ and, for } \mathbf{u} \in \mathcal{D}(A_s),\ A_s\mathbf{u} = \nu\Pi\Delta\mathbf{u}. \tag{4.2.4}$$

We also introduce the space $\mathbf{V}^s_{n,\Gamma_d}(\Omega) = \mathbf{V}^0_{n,\Gamma_d}(\Omega) \cap \mathbf{H}^s(\Omega)$ for $s \geq 0$.

To describe the Dirichlet boundary condition on $\Gamma_{s,0}$ set

$$\begin{aligned} \mathcal{L}^2(\Gamma_{s,0}) &= \{0\} \times L^2(\Gamma_{s,0}), & \mathcal{H}^{3/2}_{00}(\Gamma_{s,0}) &= \{0\} \times H^{3/2}_{00}(\Gamma_{s,0}), \\ \mathcal{H}^{\kappa}(\Gamma_{s,0}) &= \{0\} \times H^{\kappa}(\Gamma_{s,0}), & \mathcal{H}^{\kappa}_0(\Gamma_{s,0}) &= \{0\} \times H^{\kappa}_0(\Gamma_{s,0}) \text{ for } \kappa \geq 0. \end{aligned}$$

For $\kappa \geq 0$, the dual space of $\mathcal{H}^{\kappa}(\Gamma_{s,0})$ with $\mathcal{L}^2(\Gamma_{s,0})$ as pivot space is denoted by $(\mathcal{H}^{\kappa}(\Gamma_{s,0}))'$.

If X is a space of functions and $\rho \geq 0$ we set

$$\mathcal{C}^{\rho}_{\sharp}([0,T];X) := \{v_{|[0,T]} \mid v \in \mathcal{C}^{\rho}(\mathbb{R};X) \text{ is } T\text{-periodic}\}.$$

4.2.2 System in the periodic domain $\Omega_{\eta_\pi(t)}$

Let $\rho \in (0,1)$ and

$$(\boldsymbol{\omega_1}, \omega_2) \in \Big(\mathcal{C}^{\rho}_{\sharp}([0,T];\mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\rho}_{\sharp}([0,T];\mathbf{H}^{-1/2}(\Gamma_i))\Big) \times \mathcal{C}^{\rho}_{\sharp}([0,T];H^{1/2}(\Gamma_o)).$$

The regularity space for the beam equation is denoted by

$$\mathcal{C}^{\rho}_{\text{beam}} := \mathcal{C}^{\rho}_{\sharp}([0,T];H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap \mathcal{C}^{1+\rho}_{\sharp}([0,T];H^2_0(\Gamma_s)) \cap \mathcal{C}^{2+\rho}_{\sharp}([0,T];L^2(\Gamma_s)).$$

Let $(\mathbf{u}_\pi, p_\pi, \eta_\pi) \in \Big(\mathcal{C}^{1+\rho}_{\sharp}([0,T];\mathbf{H}^2(\Omega)) \cap \mathcal{C}^{2+\rho}_{\sharp}([0,T];\mathbf{L}^2(\Omega))\Big) \times \mathcal{C}^{1+\rho}_{\sharp}([0,T];H^1(\Omega)) \times \mathcal{C}^{\rho}_{\text{beam}}$ be a periodic solution of (4.1.1), that is, a solution to

$$\begin{aligned} &\mathbf{u}_{\pi,t} + (\mathbf{u}_\pi \cdot \nabla)\mathbf{u}_\pi - \operatorname{div}\sigma(\mathbf{u}_\pi, p_\pi) = 0,\ \operatorname{div}\mathbf{u}_\pi = 0 \text{ in } Q^{\eta_\pi}_T, \\ &\mathbf{u}_\pi(x,y,t) = \eta_{\pi,t}(x,1,t)\mathbf{e}_2 \text{ for } (x,y,t) \in \Sigma^{\eta_\pi}_T, \\ &\mathbf{u}_\pi = \boldsymbol{\omega}_1 \text{ on } \Sigma^i_T, \\ &u_{\pi,2} = 0 \text{ and } p_\pi = \omega_2 \text{ on } \Sigma^o_T, \\ &\mathbf{u}_\pi = 0 \text{ on } \Sigma^b_T,\ \mathbf{u}_\pi(0) = \mathbf{u}_\pi(T) \text{ in } \Omega_{\eta_\pi(0)}, \\ &\eta_{\pi,tt} - \beta\eta_{\pi,xx} - \gamma\eta_{\pi,txx} + \alpha\eta_{\pi,xxxx} = -J_{\eta_\pi(t)}\mathbf{e}_2 \cdot \sigma(\mathbf{u}_\pi, p_\pi)_{|\Gamma_{\eta_\pi(t)}}\mathbf{n}_{\eta_\pi(t)} \text{ on } \Sigma^s_T, \\ &\eta_\pi = 0 \text{ and } \eta_{\pi,x} = 0 \text{ on } \{0,L\} \times (0,T), \\ &\eta_\pi(0) = \eta_\pi(T) \text{ and } \eta_{\pi,t}(0) = \eta_{\pi,t}(T) \text{ in } \Gamma_s, \end{aligned} \tag{4.2.5}$$

and $(\mathbf{u}, p, \eta)$ be a 'perturbed' solution i.e. a solution to (4.1.1) where $(\mathbf{u}^0, \eta^0_1, \eta^0_2)$ is in a neighbourhood of $(\mathbf{u}_\pi(0), \eta_\pi(0), \eta_{\pi,t}(0)) =: (\mathbf{u}^0_\pi, \eta^0_{\pi,1}, \eta^0_{\pi,2})$.

The Hölder regularity in time of the periodic solution plays a crucial role in the stabilization of the system. Indeed, after linearization, the underlying dynamics involve the periodic solution and depend on time. In order to study the linear system we rewrite it as a matrix evolution equation driven by an operator $\mathcal{A}(t)$ with a non constant domain. Techniques in [4] are then used to prove the existence of a parabolic evolution operator. We shall see that it requires some assumption on ρ. For our analysis we fix $\rho \in (0,1)$ and we postpone the precise assumption to Section 5. Finally remark that additional time regularity is required on $\mathbf{u}_\pi$ compared to the regularity of the periodic solution obtained in Chapter 3. This is due to the non-zero divergence that appears when the perturbed solution is written in the domain of the periodic one. However, the periodic solution constructed in Chapter 3 has a small $\mathbf{u}_\pi$, an assumption that is not required in our analysis here.

To write the difference between $(\mathbf{u}, p, \eta)$ and $(\mathbf{u}_\pi, p_\pi, \eta_\pi)$ we perform the change of variables

$$\mathcal{T}_{\eta_\pi}(t) : \begin{cases} \Omega_{\eta(t)} \longrightarrow \Omega_{\eta_\pi(t)} \\ (x,y) \mapsto (x,s) = \left(x, \frac{1+\eta_\pi(t,x)}{1+\eta(x,t)} y\right). \end{cases}$$

To obtain arbitrary exponential decay after control, we introduce the parameter $\omega > 0$ and the new variables

$$\begin{aligned}
&\hat{\mathbf{u}}(x,s,t) = e^{\omega t}[\mathbf{u}(\mathcal{T}_{\eta_\pi}(t)^{-1}(x,s),t) - \mathbf{u}_\pi(x,s,t)],\\
&\hat{p}(x,s,t) = e^{\omega t}[p(\mathcal{T}_{\eta_\pi}(t)^{-1}(x,s),t) - p_\pi(x,s,t)],\\
&\hat{\eta}_1(x,t) = e^{\omega t}[\eta(x,t) - \eta_\pi(x,t)], \;\; \hat{\eta}_2(x,t) = e^{\omega t}[\eta_t(x,t) - \eta_{\pi,t}(x,t)],\\
&\hat{\mathbf{u}}_c(x,t) = e^{\omega t}\mathbf{u}_c(x,t).
\end{aligned}$$

The quadruplet $(\hat{\mathbf{u}}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2)$ satisfies the system

$$\begin{aligned}
&\hat{\mathbf{u}}_t + (\mathbf{u}_\pi \cdot \nabla)\hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \nabla)\mathbf{u}_\pi - \nu\Delta\hat{\mathbf{u}} + \nabla\hat{p} - \hat{A}_1\hat{\eta}_1 - \hat{A}_2\hat{\eta}_2 - \omega\hat{\mathbf{u}}\\
&\qquad = e^{-\omega t}F(\hat{\mathbf{u}}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2) \;\text{ in } Q_\infty^{\eta_\pi},\\
&\text{div } \hat{\mathbf{u}} = \hat{A}_3\hat{\eta}_1 + e^{-\omega t}\text{div } \mathbf{w}(\hat{\mathbf{u}}, \hat{\eta}_1) \;\text{ in } Q_\infty^{\eta_\pi},\\
&\hat{\mathbf{u}} = \hat{\eta}_2\mathbf{e}_2 \;\text{ on } \Sigma_\infty^{\eta_\pi},\\
&\hat{\mathbf{u}} = \hat{\mathbf{u}}_c \;\text{ on } \Sigma_\infty^i,\\
&\hat{u}_2 = 0 \;\text{ and }\; \hat{p} = 0, \;\text{ on } \Sigma_\infty^o,\\
&\hat{\mathbf{u}} = 0 \;\text{ on } \Sigma_\infty^b, \;\; \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \;\text{ in } \Omega_{\eta_\pi(0)}\\
&\hat{\eta}_{1,t} - \omega\hat{\eta}_1 - \hat{\eta}_2 = 0 \text{ on } \Sigma_\infty^s,\\
&\hat{\eta}_{2,t} - \omega\hat{\eta}_2 - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - \hat{A}_4\hat{\eta}_1\\
&\qquad = -J_{\eta_\pi(t)}\mathbf{e}_2 \cdot \sigma(\hat{\mathbf{u}}, \hat{p})_{|\Gamma_{\eta_\pi(t)}}\mathbf{n}_{\eta_\pi(t)} + e^{-\omega t}G(\hat{\mathbf{u}}, \hat{\eta}_1)_{|\Gamma_{\eta_\pi(t)}} \;\text{ on } \Sigma_\infty^s,\\
&\hat{\eta}_1 = 0 \;\text{ and }\; \hat{\eta}_{1,x} = 0 \;\text{ on } \{0,L\} \times (0,\infty),\\
&\hat{\eta}_1(0) = \hat{\eta}_1^0 \;\text{ and }\; \hat{\eta}_2(0) = \hat{\eta}_2^0 \;\text{ in } \Gamma_s,
\end{aligned} \tag{4.2.6}$$

where $(\hat{\mathbf{u}}^0, \hat{\eta}_1^0, \hat{\eta}_2^0) = (\mathbf{u}^0 \circ \mathcal{T}_{\eta_\pi}(0)^{-1} - \mathbf{u}_\pi^0, \eta_1^0 - \eta_{\pi,1}^0, \eta_2^0 - \eta_{\pi,2}^0)$. Using the notations

$$\tilde{\eta}_1 = e^{\omega t}\left(\frac{\eta_1 - \eta_\pi}{1+\eta_\pi}\right) = \frac{\hat{\eta}_1}{1+\eta_\pi}, \;\; \tilde{\eta}_2 = e^{\omega t}\left(\frac{\eta_1 - \eta_\pi}{1+\eta_\pi}\right)_t = \frac{\hat{\eta}_2}{1+\eta_\pi} - \frac{\eta_{\pi,t}\hat{\eta}_1}{(1+\eta_\pi)^2},$$

the additional linear terms in (4.2.6) are

$$\begin{aligned}
\hat{A}_1\hat{\eta}_1 &= -2s\tilde{\eta}_{1,x}\mathbf{u}_{\pi,xs} - s\tilde{\eta}_{1,xx}\mathbf{u}_{\pi,s} - 2\tilde{\eta}_1\mathbf{u}_{\pi,ss} + s\tilde{\eta}_{1,x}p_{\pi,s}\mathbf{e}_1 + \tilde{\eta}_1 p_{\pi,s}\mathbf{e}_2 \\
&\quad - s u_{\pi,1}\tilde{\eta}_{1,x}\mathbf{u}_{\pi,s} + u_{\pi,2}\tilde{\eta}_1\mathbf{u}_{\pi,s} - s\frac{\eta_{\pi,t}\hat{\eta}_1}{(1+\eta_\pi)^2}\mathbf{u}_{\pi,s} \\
\hat{A}_2\hat{\eta}_2 &= s\frac{\hat{\eta}_2}{1+\eta_\pi}\mathbf{u}_{\pi,s} \\
\hat{A}_3\hat{\eta}_1 &= \operatorname{div}(-\tilde{\eta}_1 u_{\pi,1}\mathbf{e}_1 + s\tilde{\eta}_{1,x}u_{\pi,1}\mathbf{e}_2) = -\tilde{\eta}_1 u_{\pi,1,x} + s\tilde{\eta}_{1,x}u_{\pi,1,s} \\
\hat{A}_4\hat{\eta}_1 &= \nu\hat{\eta}_{1,x}u_{\pi,2,x} + 2\nu\tilde{\eta}_1 u_{\pi,2,s} + \nu\hat{\eta}_{1,x}u_{\pi,1,s} - \nu\tilde{\eta}_1\eta_{\pi,x}u_{\pi,1,s} - \nu s\eta_{\pi,x}\tilde{\eta}_{1,x}u_{\pi,2,s}.
\end{aligned}$$

The genuinely non-linear terms are

$$\mathbf{w}(\hat{\mathbf{u}}, \hat{\eta}_1) = -\tilde{\eta}_1\hat{u}_1\mathbf{e}_1 + s\tilde{\eta}_{1,x}\hat{u}_1\mathbf{e}_1,$$

$$\begin{aligned}
F(\hat{\mathbf{u}}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2) &= s\tilde{\eta}_2\hat{\mathbf{u}} - \frac{se^{-\omega t}\tilde{\eta}_1\tilde{\eta}_2}{1+e^{-\omega t}\tilde{\eta}_1}\hat{\mathbf{u}}_s - \frac{s\tilde{\eta}_1\tilde{\eta}_2}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,s} - 2s\tilde{\eta}_{1,x}\hat{\mathbf{u}}_{xs} + \frac{2se^{-\omega t}\tilde{\eta}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{\mathbf{u}}_{xs} \\
&+ \frac{2s\tilde{\eta}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,xs} + e^{-\omega t}\left(s\frac{\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\right)^2\hat{\mathbf{u}}_{ss} + \left(s\frac{\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\right)^2\mathbf{u}_{\pi,ss} - s\tilde{\eta}_{1,xx}\hat{\mathbf{u}}_s \\
&+ se^{-\omega t}\frac{\tilde{\eta}_1\tilde{\eta}_{1,xx}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{\mathbf{u}}_s + s\frac{\tilde{\eta}_1\tilde{\eta}_{1,xx}}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,s} - s\frac{e^{-\omega t}\tilde{\eta}_{1,x}^2}{(1+e^{-\omega t}\tilde{\eta}_1)^2}\hat{\mathbf{u}}_s - s\frac{\tilde{\eta}_{1,x}^2}{(1+e^{-\omega t}\tilde{\eta}_1)^2}\mathbf{u}_{\pi,s} \\
&- 2\tilde{\eta}_1\hat{\mathbf{u}}_{ss} + \frac{3e^{-\omega t}\tilde{\eta}_1^2 + 2e^{-2\omega t}\tilde{\eta}_1^3}{(1+e^{-\omega t}\tilde{\eta}_1)^2}\hat{\mathbf{u}}_{ss} + \frac{3\tilde{\eta}_1^2 + 2e^{-\omega t}\tilde{\eta}_1^3}{(1+e^{-\omega t}\tilde{\eta}_1)^2}\mathbf{u}_{\pi,ss} + s\tilde{\eta}_{1,x}\hat{p}_s\mathbf{e}_1 - \frac{se^{-\omega t}\tilde{\eta}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{p}_s\mathbf{e}_1 \\
&- \frac{s\tilde{\eta}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}p_{\pi,s}\mathbf{e}_1 + \tilde{\eta}_1\hat{p}_s\mathbf{e}_2 - \frac{e^{-\omega t}\tilde{\eta}_1^2}{1+e^{-\omega t}\tilde{\eta}_1}\hat{p}_s\mathbf{e}_2 - \frac{\tilde{\eta}_1^2}{1+e^{-\omega t}\tilde{\eta}_1}p_{\pi,s}\mathbf{e}_2 - \frac{u_{\pi,2}\tilde{\eta}_1^2}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,s} \\
&+ \frac{su_{\pi,1}\tilde{\eta}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,s} - (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}} - s\frac{e^{-\omega t}\hat{u}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{\mathbf{u}}_s - s\frac{\hat{u}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,s} - s\frac{u_{\pi,1}\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{\mathbf{u}}_s \\
&+ \frac{e^{-\omega t}\hat{u}_2\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}_1}\hat{\mathbf{u}}_s + \frac{\hat{u}_2\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}_1}\mathbf{u}_{\pi,s} + \frac{u_{\pi,2}\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}}\hat{\mathbf{u}}_s,
\end{aligned}$$

and

$$\begin{aligned}
G(\hat{\mathbf{u}}, \hat{\eta}_1) &= \nu\hat{\eta}_{1,x}\hat{u}_{2,x} - \frac{2\nu\tilde{\eta}_1^2}{1+e^{-\omega t}\tilde{\eta}_1}u_{\pi,2,s} + \frac{2\nu\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}_1}\hat{u}_{2,s} + \frac{\nu\eta_{\pi,x}\tilde{\eta}_1^2}{1+e^{-\omega t}\tilde{\eta}_1}u_{\pi,1,s} \\
&+ \nu\hat{\eta}_{1,x}\hat{u}_{1,s} - \frac{\nu e^{-\omega t}\hat{\eta}_{1,x}\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}_1}\hat{u}_{1,s} - \frac{\nu\hat{\eta}_{1,x}\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}_1}u_{\pi,1,s} - \frac{\nu\eta_{\pi,x}\tilde{\eta}_1}{1+e^{-\omega t}\tilde{\eta}_1}\hat{u}_{1,s} \\
&+ \frac{\nu\eta_{\pi,x}\tilde{\eta}_1\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}u_{\pi,2,s} - \frac{\nu se^{-\omega t}\hat{\eta}_{1,x}\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{u}_{2,s} - \frac{\nu s\hat{\eta}_{1,x}\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}u_{\pi,2,s} - \frac{\nu s\eta_{\pi,x}\tilde{\eta}_{1,x}}{1+e^{-\omega t}\tilde{\eta}_1}\hat{u}_{2,s}.
\end{aligned}$$

Remark 4.2.1. The coefficients of the linear operator $\hat{A}_4$, which involve $\mathbf{u}_\pi$, must be understood in the sense of the trace on $\Gamma_{\eta_\pi(t)}$ parametrized on Γ_s. To make things explicit

$$\begin{aligned}\hat{A}_4\hat{\eta}_1 &= \nu\hat{\eta}_{1,x}u_{\pi,2,x}(x,1+\eta_\pi(x,t)) + 2\nu\tilde{\eta}_1 u_{\pi,2,s}(x,1+\eta_\pi(x,t)) \\ &\quad + \nu\hat{\eta}_{1,x}u_{\pi,1,s}(x,1+\eta_\pi(x,t)) - \nu\tilde{\eta}_1\eta_{\pi,x}u_{\pi,1,s}(x,1+\eta_\pi(x,t)) \\ &\quad - \nu s\eta_{\pi,x}\tilde{\eta}_{1,x}u_{\pi,2,s}(x,1+\eta_\pi(x,t)).\end{aligned}$$

The same abuse of notations is done in Section 2.3 when the system is written in a fixed domain. That is with η_π replaced by $\eta^0_{\pi,1}$.

Keeping the same notations for $(\hat{\mathbf{u}},\hat{p},\hat{\eta}_1,\hat{\eta}_2)$ we linearize (4.2.6) near $(\mathbf{0},0,0,0)$.

$$\begin{aligned}
&\hat{\mathbf{u}}_t + (\mathbf{u}_\pi\cdot\nabla)\hat{\mathbf{u}} + (\hat{\mathbf{u}}\cdot\nabla)\mathbf{u}_\pi - \nu\Delta\hat{\mathbf{u}} + \nabla\hat{p} - \hat{A}_1\hat{\eta}_1 - \hat{A}_2\hat{\eta}_2 - \omega\hat{\mathbf{u}} = 0 \ \text{ in } Q^{\eta_\pi}_\infty,\\
&\operatorname{div}\hat{\mathbf{u}} = \hat{A}_3\hat{\eta}_1 \ \text{ in } Q^{\eta_\pi}_\infty,\\
&\hat{\mathbf{u}} = \hat{\eta}_2\mathbf{e}_2 \ \text{ on } \Sigma^{\eta_\pi}_\infty,\\
&\hat{\mathbf{u}} = \hat{\mathbf{u}}_c \ \text{ on } \Sigma^i_\infty,\\
&\hat{u}_2 = 0 \ \text{ and } \ \hat{p} = 0, \ \text{ on } \Sigma^o_\infty,\\
&\hat{\mathbf{u}} = 0 \ \text{ on } \Sigma^b_\infty, \ \ \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \ \text{ in } \Omega_{\eta_\pi(0)},\\
&\hat{\eta}_{1,t} - \omega\hat{\eta}_1 - \hat{\eta}_2 = 0 \text{ on } \Sigma^s_\infty,\\
&\hat{\eta}_{2,t} - \omega\hat{\eta}_2 - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - \hat{A}_4\hat{\eta}_1\\
&\qquad = -J_{\eta_\pi(t)}\mathbf{e}_2\cdot\sigma(\hat{\mathbf{u}},\hat{p})_{|\Gamma_{\eta_\pi(t)}}\mathbf{n}_{\eta_\pi(t)} \ \text{ on } \Sigma^s_\infty,\\
&\hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\hat{\eta}_1(0) = \hat{\eta}^0_1 \ \text{ and } \ \hat{\eta}_2(0) = \hat{\eta}^0_2 \ \text{ in } \Gamma_s.
\end{aligned}\tag{4.2.7}$$

4.2.3 System in the fixed domain $\Omega_{\eta_\pi(0)}$

To study the linear system (4.2.7) in a fixed domain we perform another change of variables:

$$\mathcal{T}_{\eta^0_{\pi,1}}(t) : \begin{cases} \Omega_{\eta_\pi(t)} \longrightarrow \Omega_{\eta^0_{\pi,1}} \\ (x,s)\mapsto(x,z) = \left(x, \frac{1+\eta^0_{\pi,1}(x)}{1+\eta_\pi(x,t)}s\right), \end{cases}$$

where we recall that $\eta^0_{\pi,1} = \eta_\pi(0)$. To simplify the notations, set $\Omega_{\pi,0} = \Omega_{\eta^0_{\pi,1}}$, $\Gamma_{\pi,0} = \Gamma_{\eta^0_{\pi,1}}$, $Q^{\pi,0}_T = Q^{\eta^0_{\pi,1}}_T$ and $\Sigma^{\pi,0}_T = \Sigma^{\eta^0_{\pi,1}}_T$. Introducing the new unknowns

$$\begin{aligned}
&\mathbf{v}(x,z,t) = \hat{\mathbf{u}}(\mathcal{T}_{\eta^0_{\pi,1}}(t)^{-1}(x,z),t), \ \ q(x,z,t) = \hat{p}(\mathcal{T}_{\eta^0_{\pi,1}}(t)^{-1}(x,z),t),\\
&\overline{\mathbf{u}}_\pi(x,z,t) = \mathbf{u}_\pi(\mathcal{T}_{\eta^0_{\pi,1}}(t)^{-1}(x,z),t), \ \ \overline{p}_\pi(x,z,t) = p_\pi(\mathcal{T}_{\eta^0_{\pi,1}}(t)^{-1}(x,z),t),
\end{aligned}$$

the linear system (4.2.7) becomes

$$
(4.2.8)\quad
\begin{aligned}
&\mathbf{v}_t + (\overline{\mathbf{u}}_\pi \cdot \nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\overline{\mathbf{u}}_\pi + C_\pi \mathbf{v}_z - \nu\Delta\mathbf{v} + \nabla q - A_{1,1}\hat{\eta}_1 - A_2\hat{\eta}_2 - \omega\mathbf{v}\\
&\qquad = L_\pi(\mathbf{v},q) + A_{1,2}\hat{\eta}_1 \ \text{ in } Q_\infty^{\pi,0},\\
&\operatorname{div}\mathbf{v} = A_3\hat{\eta}_1 \ \text{ in } Q_\infty^{\pi,0},\\
&\mathbf{v} = \hat{\eta}_2\mathbf{e}_2 \ \text{ on } \Sigma_\infty^{\pi,0},\\
&\mathbf{v} = \mathbf{v}_c \ \text{ on } \Sigma_\infty^{i},\\
&v_2 = 0 \ \text{ and } \ q = 0, \ \text{ on } \Sigma_\infty^{o},\\
&\mathbf{v} = 0 \ \text{ on } \Sigma_\infty^{b}, \ \ \mathbf{v}(0) = \mathbf{v}^0 \ \text{ in } \Omega_{\pi,0},\\
&\hat{\eta}_{1,t} - \omega\hat{\eta}_1 - \hat{\eta}_2 = 0 \text{ on } \Sigma_\infty^{s},\\
&\hat{\eta}_{2,t} - \omega\hat{\eta}_2 - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1\\
&\qquad = -J_{\eta^0_{\pi,1}}\mathbf{e}_2\cdot\sigma(\mathbf{v},q)_{|\Gamma_{\eta^0_{\pi,1}}}\mathbf{n}_{\eta^0_{\pi,1}} + A_{4,2}\hat{\eta}_1 + C\mathbf{v} \ \text{ in } \Sigma_\infty^{s},\\
&\hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\hat{\eta}_1(0) = \hat{\eta}_1^0 \ \text{ and } \ \hat{\eta}_2(0) = \hat{\eta}_2^0 \ \text{ in } \Gamma_s.
\end{aligned}
$$

with $\mathbf{v}^0 = \hat{\mathbf{u}}^0$, $\mathbf{v}_c = \hat{\mathbf{u}}_c$ and

$$
\begin{aligned}
&\tilde{\eta}_\pi = \frac{\eta_\pi - \eta_\pi^0}{1+\eta_\pi^0}\\
&C_\pi = \frac{z\tilde{\eta}_{\pi,t}}{1+\tilde{\eta}_\pi}\\
&A_{1,1}\hat{\eta}_1 = -2z\tilde{\eta}_{1,x}\overline{\mathbf{u}}_{\pi,xz} - z\tilde{\eta}_{1,xx}\overline{\mathbf{u}}_{\pi,z} - 2\tilde{\eta}_1\overline{\mathbf{u}}_{\pi,zz} + z\tilde{\eta}_{1,x}\overline{p}_{\pi,z}\mathbf{e}_1\\
&\qquad + \tilde{\eta}_1\overline{p}_{\pi,s}\mathbf{e}_2 - z\frac{\eta_{\pi,t}\hat{\eta}_1}{(1+\eta_\pi)^2}\overline{\mathbf{u}}_{\pi,z} + \overline{u}_{\pi,2}\tilde{\eta}_1\overline{\mathbf{u}}_{\pi,z} - z\frac{\eta_{\pi,t}\hat{\eta}_1}{(1+\eta_\pi)^2}\overline{\mathbf{u}}_{\pi,z}\\
&A_{1,2}\hat{\eta}_1 = \frac{2z\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\tilde{\eta}_{1,x}\overline{\mathbf{u}}_{\pi,z} + \frac{2z^2\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\tilde{\eta}_{1,x}\overline{\mathbf{u}}_{\pi,zz} + 2\frac{\tilde{\eta}_\pi + \tilde{\eta}_\pi^2}{(1+\tilde{\eta}_\pi)^2}\tilde{\eta}_1\overline{\mathbf{u}}_{\pi,zz}\\
&\qquad - \frac{\tilde{\eta}_\pi\tilde{\eta}_1}{1+\tilde{\eta}_\pi}\overline{p}_{\pi,z}\mathbf{e}_2 - \frac{\overline{u}_{\pi,2}\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}\tilde{\eta}_1\overline{\mathbf{u}}_{\pi,z}\\
&A_2\hat{\eta}_2 = z\frac{\hat{\eta}_2}{1+\eta_\pi}\mathbf{u}_{\pi,z}\\
&A_3\hat{\eta}_1 = \operatorname{div}(-\tilde{\eta}_1\overline{u}_{\pi,1}\mathbf{e}_1 + z\tilde{\eta}_{1,x}\overline{u}_{\pi,1}\mathbf{e}_2) = -\tilde{\eta}_1\overline{u}_{\pi,1,x} + z\tilde{\eta}_{1,x}\overline{u}_{\pi,1,z}\\
&A_{4,1}\hat{\eta}_1 = \nu\hat{\eta}_{1,x}\overline{u}_{\pi,2,x} + 2\nu\tilde{\eta}_1\overline{u}_{\pi,2,z} + \nu\hat{\eta}_{1,x}\overline{u}_{\pi,1,z} - \nu\tilde{\eta}_1\eta_{\pi,x}\overline{u}_{\pi,1,z} - \nu z\eta_{\pi,x}\tilde{\eta}_{1,x}\overline{u}_{\pi,2,z}\\
&A_{4,2}\hat{\eta}_1 = -\frac{\nu z\hat{\eta}_{1,x}\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\overline{u}_{\pi,2,z} - \frac{2\nu\tilde{\eta}_\pi\tilde{\eta}_1}{1+\tilde{\eta}_\pi}\overline{u}_{\pi,2,z} - \frac{\nu\hat{\eta}_{1,x}\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}\overline{u}_{\pi,1,z} + \frac{\nu\tilde{\eta}_1\eta_{\pi,x}\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}\overline{u}_{\pi,1,z},
\end{aligned}
$$

and

$$
C\mathbf{v} = \nu(\eta_\pi - \eta^0_{\pi,1})_x v_{2,x} + \nu(\eta_\pi - \eta^0_{\pi,1})_x v_{1,z} - \frac{z\nu\eta_{\pi,x}\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}v_{2,z} + \frac{2\nu\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}v_{2,z} - \frac{\nu\eta_{\pi,x}\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}v_{1,z},
$$

$$L_\pi(\mathbf{v},q) = \frac{\overline{u}_{\pi,1} z\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\mathbf{v}_z + \frac{\overline{u}_{\pi,2}\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}\mathbf{v}_z + \frac{v_1 z\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\overline{\mathbf{u}}_{\pi,z} + \frac{v_2\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}\overline{\mathbf{u}}_{\pi,z} + \frac{2\tilde{\eta}_\pi+\tilde{\eta}_\pi^2}{(1+\tilde{\eta}_\pi)^2}\mathbf{v}_{zz}$$
$$+\frac{2z\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\mathbf{v}_{xz} - \left(\frac{z\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}\right)^2\mathbf{v}_{zz} + \frac{z\tilde{\eta}_{\pi,xx}}{1+\tilde{\eta}_\pi}\mathbf{v}_z - \frac{z\tilde{\eta}_{\pi,x}}{(1+\tilde{\eta}_\pi)^2}\mathbf{v}_z + \frac{z\tilde{\eta}_{\pi,x}}{1+\tilde{\eta}_\pi}q_z + \frac{\tilde{\eta}_\pi}{1+\tilde{\eta}_\pi}q_z.$$

System (4.2.8) involves additional linear terms of higher order, and therefore resists to the standard analysis (for example, computing the spectrum). However, these terms involve the difference $\eta_\pi - \eta_\pi^0$. To solve this issue we first study a system where these higher order terms have been removed. These terms can then be re-introduced in a future study of the nonlinear system with a suitable smallness assumptions on $\eta_\pi - \eta_\pi^0$. The linear system that we study, with $\omega = 0$, is

$$\begin{aligned}
&\mathbf{v}_t + (\overline{\mathbf{u}}_\pi\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\overline{\mathbf{u}}_\pi + C_\pi\mathbf{v}_z - \nu\Delta\mathbf{v} + \nabla q - A_{1,1}\hat{\eta}_1 - A_2\hat{\eta}_2 = 0 \ \text{ in } Q_\infty^{\pi,0},\\
&\text{div }\mathbf{v} = A_3\hat{\eta}_1 \ \text{ in } Q_\infty^{\pi,0},\\
&\mathbf{v} = \hat{\eta}_2\mathbf{e}_2 \ \text{ on } \Sigma_\infty^{\pi,0},\\
&\mathbf{v} = \mathbf{v}_c \ \text{ on } \Sigma_\infty^{i},\\
&v_2 = 0 \ \text{ and } \ q = 0, \ \text{ on } \Sigma_\infty^{o},\\
&\mathbf{v} = 0 \ \text{ on } \Sigma_\infty^{b}, \ \ \mathbf{v}(0) = \mathbf{v}^0 \ \text{ in } \Omega_{\pi,0},\\
&\hat{\eta}_{1,t} = \hat{\eta}_2 \text{ on } \Sigma_\infty^{s},\\
&\hat{\eta}_{2,t} - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1\\
&\qquad = -J_{\eta_{\pi,1}^0}\mathbf{e}_2\cdot\sigma(\mathbf{v},q)_{|\Gamma_{\eta_{\pi,1}^0}}\mathbf{n}_{\eta_{\pi,1}^0} \ \text{ in } \Sigma_\infty^{s},\\
&\hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\hat{\eta}_1(0) = \hat{\eta}_1^0 \ \text{ and } \ \hat{\eta}_2(0) = \hat{\eta}_2^0 \ \text{ in } \Gamma_s.
\end{aligned}\tag{4.2.9}$$

4.3 Time-dependent evolution equations

4.3.1 The parabolic evolution operator

In order to study (4.2.9) we apply the semigroups techniques used in Chapter 2, which were introduced in [56] for different boundary conditions. The present case is more complicated as the linear system presents time-dependent coefficients. Moreover, as we will see in Section 4.5, the domain of the family of operators governing the linear system also depends on time. In this section we use the notations introduced in Section 1.3.4 in Chapter 1. We present two different situations:

1. A family of operators $(A(t), \mathcal{D}(A(t)))_{0\le t\le T}$ where $\mathcal{D}(A(t)) = \mathcal{D}$ is independent of t.

2. A family of operators $(A(t), \mathcal{D}(A(t)))_{0\le t\le T}$ where $\mathcal{D}(A(t))$ depends on t.

For each case, we describe the functional framework considered to construct the parabolic evolution operator associated to $(A(t), \mathcal{D}(A(t)))_{0\le t\le T})$.

Let (E_0, E_1) be a pair of densely embedded Banach spaces $E_1 \overset{d}{\hookrightarrow} E_0$. We introduce the notion of parabolic evolution operator. For $T > 0$ we use the following notations:

$$\Delta_T := \{(t,s) \mid 0 \le s \le t \le T\} \text{ and } \dot{\Delta}_T := \{(t,s) \mid 0 \le s < t \le T\}.$$

We use the notation $\mathcal{L}_s(E_1, E_0)$ to denote the space of bounded linear operator from E_1 to E_0 equipped with the strong operator topology.

Definition 4.3.1. Suppose that $F \hookrightarrow E_0$ and let $(A(t), \mathcal{D}(A(t)))_{0\le t\le T}$ be a family of closed linear operator such that $\mathcal{D}(A(t)) \subset F$. A map $U : \Delta_T \to \mathcal{L}(E_0)$ is said to be a parabolic evolution operator for $(A(t), \mathcal{D}(A(t)))_{0\le t\le T}$ with regularity space F if

(U_1) $U \in \mathcal{C}(\Delta_T; \mathcal{L}_s(E_0)) \cap \mathcal{C}(\dot{\Delta}_T; \mathcal{L}(E_0, F))$
and $\mathrm{Range}(U(t,s)) \subset \mathcal{D}(A(t)) \subset F$ for all $(t,s) \in \dot{\Delta}_T$.

(U_2) $U(t,t) = I$, $U(t,r) = U(t,s)U(s,r)$ for all $0 \le r \le s \le t \le T$.

(U_3) $[(t,s) \mapsto A(t)U(t,s)] \in \mathcal{C}(\dot{\Delta}_T; \mathcal{L}(E_0))$ and
$\sup_{(t,s)\in\dot{\Delta}_T} (t-s) \|A(t)U(t,s)\|_{\mathcal{L}(E_0)} < +\infty.$

(U_4) $U(\cdot, s) \in \mathcal{C}^1((s,T]; \mathcal{L}(E_0))$ for each $s \in [0,T)$ and, for all $t \in (s,T]$,
$\partial_1 U(t,s) = -A(t)U(t,s)$,
$U(t,\cdot) \in \mathcal{C}^1([0,t); \mathcal{L}_s(F, E_0))$ for each $t \in (0,T]$ and, for all $s \in [0,t)$,
$\partial_2 U(t,s) \supset U(t,s)A(t)$.

We can now state sufficient assumptions on $(A(t), \mathcal{D}(A(t)))_{t\ge0}$ to ensure the existence of the evolution operator when the domain is constant. We assume that $(A(t), \mathcal{D}(A(t)))_{t\ge0}$ satisfies the following properties:

(A_1) $\mathcal{D}(A(t)) = E_1$ for all $t \in [0,T]$,

(A_2) There exists $\lambda_0 \in \mathbb{R}$ and $M > 0$ such that $A(t) \in \mathfrak{A}_{M,\lambda_0}(E_0)$ for all $t \in [0,T]$.

(A_3) There exists a constant $\rho \in (0,1)$, such that $A(\cdot) \in \mathcal{C}^\rho([0,T]; \mathcal{L}(E_1, E_0))$.

The following theorem can be deduced using [4, Section III]; see also [34].

Theorem 4.3.1. Suppose that the family of closed linear operators $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{0\le t\le T}$ satisfies Assumptions (A_1)–(A_3). Then there exists a unique parabolic evolution operator U with regularity subspace E_1.

The second case (2), that is when the domain $\mathcal{D}(A(t))$ is not constant, was investigated in [4, Section IV]. One of the idea is to consider a family of operators $(A(t), \mathcal{D}(A(t)))_{t\geq 0}$ where the domain of $A(t)$ may vary with t but a suitable interpolation space is constant. The author then proves the existence of an evolution operator where the densely injected Banach couple (E_1, E_0) has been replaced by (E_α, E_0) with E_α a interpolation space between $\mathcal{D}(A(t))$ and E_0.

Let us give some details. Given $A : [0,T] \to \mathfrak{A}_{M,\lambda_0}(E_0)$ we set $E_1(A(t)) := \mathcal{D}(A(t))$ endowed with the graph norm of $A(t)$. Remark that $A(t) \in \mathcal{H}(E_1(A(t)), E_0)$. We assume that:

(B$_1$) There exists constants $\theta \in (0,1)$, $C \geq 1$, and a Banach space E_θ with norm $\|\cdot\|_\theta$ such that, for all $t \in [0,T]$,

$$E_\theta(A(t)) := [E_1(A(t)), E_0]_{1-\theta} = E_\theta,$$

and

$$C^{-1}\|x\|_\theta \leq \|x\|_{E_\theta(A(t))} \leq C\|x\|_\theta, \quad x \in E_\theta.$$

(B$_2$) There exists $\rho \in (1-\theta, 1)$ such that $(\lambda_0 - A)^{-1} \in \mathcal{C}^\rho([0,T]; \mathcal{L}(E_0, E_\theta))$.

The following theorem is proved in [4].

Theorem 4.3.2. Suppose that $A : [0,T] \to \mathfrak{A}_{M,\lambda_0}(E_0)$ satisfies Assumptions (B_1)–(B_2). Then there exists a unique parabolic evolution operator U with regularity subspace E_θ.

Remark 4.3.1. In [4], the family of operators $A : [0,T] \to \mathfrak{A}_{M,\lambda_0}(E_0)$ is chosen with $\lambda_0 = 0$. This choice is made for simplicity. Indeed setting $A_{\lambda_0}(t) := A(t) - \lambda_0 I$ we see that $A_{\lambda_0}(t) \in \mathfrak{A}_{M,0}(E_0)$. Then [4, Theorem 2.3.2, Chapter IV] implies the existence of a parabolic evolution operator U_{λ_0} for A_{λ_0} and $U(t,s) := e^{-\lambda_0(t-s)}U_{\lambda_0}(t,s)$ is the evolution operator for $(A(t), \mathcal{D}(A(t)))_{0\leq t\leq T}$.

4.3.2 A perturbation result

Theorem 4.3.3. For $\lambda_0 \in \mathbb{R}$ and $M > 0$ suppose that $A : [0,T] \to \mathfrak{A}_{M,\lambda_0}(E_0)$. For $\kappa \in (0,1)$ suppose that $B : [0,T] \to \mathcal{L}([E_1(A(t)), E_0]_{1-\kappa}, E_0)$ with

$$\sup_{t\in[0,T]} \|B(t)\|_{\mathcal{L}([E_1(A(t)),E_0]_{1-\kappa},E_0)} < +\infty.$$

Then there exists $\lambda_0' \in \mathbb{R}$ and $M' > 0$ such that $A + B : [0,T] \to \mathfrak{A}_{M',\lambda_0'}(E_0)$.

Proof. For all $\lambda \in \mathbb{C}$ such that $\operatorname{Re} z \geq \lambda_0$ consider the resolvent equation with $\mathbf{z} \in E_0$:

$$\lambda\mathbf{u} - A(t)\mathbf{u} - B(t)\mathbf{u} = \mathbf{z}. \tag{4.3.1}$$

Setting $\mathbf{v} = \lambda\mathbf{u} - A(t)\mathbf{u}$ the previous equation becomes

$$\mathbf{v} = B(t)R(\lambda, A(t))\mathbf{v} + \mathbf{z}, \tag{4.3.2}$$

where $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$. Consider the interpolation inequality [43, Corollary 1.2.7]

$$\|\cdot\|_{[E_1(A(t)),E_0]_{1-\kappa}} \leq C(\kappa)\, \|\cdot\|^{\kappa}_{E_1(A(t))}\, \|\cdot\|^{1-\kappa}_{E_0}, \tag{4.3.3}$$

We set $C_B = \sup_{t\in[0,T]} \|B(t)\|_{\mathcal{L}([E_1(A(t)),E_0]_{1-\kappa},E_0)}$. Using (4.3.3) and the resolvent estimate on $A(t)$ we obtain, for all $t \in [0, T]$,

$$\begin{aligned}
\|B(t)R(\lambda, A(t))\|_{\mathcal{L}(E_0)} &\leq \|B(t)\|_{\mathcal{L}([E_1(A(t)),E_0]_{1-\kappa},E_0)}\, \|R(\lambda, A(t))\|_{\mathcal{L}(E_0,[E_1(A(t)),E_0]_{1-\kappa})}\\
&\leq C_B C(\kappa)\, \|R(\lambda, A(t))\|^{\kappa}_{\mathcal{L}(E_0,E_1(A(t)))}\, \|R(\lambda, A(t))\|^{1-\kappa}_{\mathcal{L}(E_0)}\\
&\leq C_B C(\kappa) \left(\frac{M}{1+|\lambda|}\right)^{1-\kappa} (\|R(\lambda, A(t))\|_{\mathcal{L}(E_0)} + \|A(t)R(\lambda, A(t))\|_{\mathcal{L}(E_0)})^{\kappa}.
\end{aligned}$$

Then, using that $A(t)R(\lambda, A(t)) = \lambda R(\lambda, A(t)) - I$ we obtain

$$\|B(t)R(\lambda, A(t))\|_{\mathcal{L}(E_0)} \leq C_B C(\kappa)\,(M+1)^{\kappa} \left(\frac{M}{1+|\lambda|}\right)^{1-\kappa}.$$

This shows that, for $\operatorname{Re}\lambda' > \lambda_0$ large enough, $\|B(t)R(\lambda', A(t))\|_{\mathcal{L}(E_0)} \leq \frac{1}{2}$. Hence (4.3.2) admits a unique solution $\mathbf{v} \in E_0$ with the estimate $\|\mathbf{v}\|_{E_0} \leq 2\,\|\mathbf{z}\|_{E_0}$. Coming back to $\mathbf{u}$ we have proved that the resolvent equation (4.3.1) admits a unique solution $\mathbf{u} = R(\lambda', A(t))\mathbf{v} \in \mathcal{D}(A(t))$ which satisfies the following estimate

$$\|\mathbf{u}\|_{E_0} \leq \frac{2M}{1+|\lambda'|}\, \|\mathbf{z}\|_{E_0}.$$

The theorem is proved with $M' = 2M$. □

4.3.3 Periodic framework and stabilisation

Consider a family $(A(t), \mathcal{D}(A(t))_{t\geq 0}$ of closed linear operators satisfying the assumptions (B_1)–(B_2) and

$$A(t+T) = A(t) \text{ for any } t \geq 0.$$

Suppose additionally that E_θ is compactly embedded in E_0. Setting

$$\Delta := \{(t, s) \in \mathbb{R}^2 \mid s \leq t\} \text{ and } \dot{\Delta} := \{(t, s) \in \mathbb{R}^2 \mid s < t\},$$

and using Theorem 4.3.2 we know that there exists a unique parabolic evolution operator $U : \Delta \to \mathcal{L}(E_0)$ satisfying (U_1)–(U_4) on Δ_{nT} for all $n \in \mathbb{N}^*$. We then introduce the Poincaré map, defined for $t \geq 0$ by

$$V(t) = U(t, t+T).$$

Let us recall some basic properties of $V(\cdot)$ [20, Proposition 6.2, Section II].

Lemma 4.3.1. The following properties hold:

- The map $V(\cdot)$ is T-periodic.
- $\sigma(V(t)) \setminus \{0\}$ is independent of $t \geq 0$.
- $\sigma_p(V(t)) \setminus \{0\}$ is independent of $t \geq 0$.

The spectrum of the Poincaré map plays a crucial role in the stabilization theory. In order to investigate the behaviour of the evolution equation $y'(t) = A(t)y(t) + f(t)$ we want to split the state space in two parts, one stable and one unstable. When the operator A does not depend on time this is done considering the eigenvalues of A with a real part smaller than 0 on one side, and larger than 0 on the other side. Here the eigenvalues of $A(t)$ may depend on time. Hence we introduce the Poincaré operator to study the evolution of y on a period T. Its spectrum does not depend on time and the splitting used in the constant case is replaced by considering the eigenvalue of V with an absolute value smaller than 1 on one side and larger that 1 on the other side.

In order to study the spectrum of V in $\mathbb{C}$ we complexify all the operators and spaces, keeping the same notations.

Lemma 4.3.2. For $t \geq 0$, $\sigma(V(t)) \setminus \{0\} = \sigma_p(V(t)) \setminus \{0\}$ and the eigenvalues of $V(t)$ have finite algebraic multiplicity. The eigenvalues of $V(t)$, denoted by $\{\lambda_j \mid j \in \mathbb{N}\}$, can be ordered such that

$$\cdots \leq |\lambda_{j+1}| \leq |\lambda_j| \leq \cdots \leq |\lambda_1|,$$

and $|\lambda_j| \to 0$ when $j \to +\infty$.

Proof. We have to prove that $V(t)$ is a compact operator. This result follows directly from Assumption (U_1), which implies that $V(t) = U(t, t+T) \in \mathcal{L}(E_0, E_\theta)$, and from the compact embedding $E_\theta \overset{c}{\hookrightarrow} E_0$. □

Recalling that $-\omega < 0$ is the decreasing rate let $N = N_\omega \in \mathbb{N}$ be such that

$$\cdots \leq |\lambda_{N+1}| < e^{-\omega T} \leq |\lambda_N| \leq \cdots \leq |\lambda_1|.$$

Without loss of generality we suppose that $|\lambda_{N+1}| < e^{-\omega T} < |\lambda_N|$. For $t \geq 0$, the unstable part of the spectrum of $V(t)$ is given by $\{\lambda_j \mid 0 \leq j \leq N\}$ and the stable part is outside the disk $D_\omega := D(0, e^{-\omega T}) = \{z \in \mathbb{C} \mid |z| < e^{-\omega T}\}$. Setting $\Gamma = \partial D_\omega$, we introduce the projections, for $t \geq 0$,

$$P_s(t) = \frac{1}{2i\pi} \int_\Gamma R(z, V(t)) dz, \quad P_u(t) = I - P_s. \tag{4.3.4}$$

The projector $P_s(t)$ is defined via a Dunford's integral. Using the functional calculus associated with Cauchy's formula, we have $P_s(t) = \mathbb{1}_{D_\omega}(V(t))$ where $\mathbb{1}_{D_\omega}$ is the indicator function of D_ω. The projection properties follow directly from the previous identity and the associated functional calculus [62, Theorem, Section VIII.7]. Moreover we have $\sigma(\mathbb{1}_{D_\omega}(V(t))) = \mathbb{1}_{D_\omega}(\sigma(V(t)))$. For $t \geq 0$, setting $X_s(t) := P_s(t)E_0$ and $X_u(t) := P_u(t)E_0$ we obtain a decomposition of E_0,

$$E_0 = X_s(t) \oplus X_u(t). \tag{4.3.5}$$

The restriction of $V(t)$ to $X_i(t)$ is denoted by $V_i(t)$ for $i \in \{s, u\}$.

Lemma 4.3.3. The following properties hold:

1. For $(t_1, t_2) \in \Delta$, $U(t_1, t_2) \in \mathcal{L}(X_s(t_1), X_s(t_2))$.
2. For $(t_1, t_2) \in \Delta$, $U(t_1, t_2) \in \mathrm{Iso}(X_u(t_1), X_u(t_2))$.
3. For $t \geq 0$, the decomposition (4.3.5) of E_0 is $V(t)$-invariant.
4. For $t \geq 0$, $\sigma(V_u(t)) = \{\lambda_j\}_{0 \leq j \leq N}$ and $\sigma(V_s(t)) = \{\lambda_j\}_{j \geq N+1}$.

Proof. The points (1)–(2) are proved in [20, Lemma 7.1, Section II]. The point (3) is a direct consequence of the previous ones, and (4) follows from the spectral mapping theorem. □

For $(t_1, t_2) \in \Delta$ and $i \in \{s, u\}$, define

$$U_i(t_1, t_2) = U(t_1, t_2)P_i(t_2),$$

We have obtained the following decomposition for the parabolic evolution operator $U : \Delta \to \mathcal{L}(E_0)$,

$$U(t_1, t_2) = U_s(t_1, t_2) \oplus U_u(t_1, t_2) : X_s(t_2) \oplus X_u(t_2) \to X_s(t_1) \oplus X_u(t_1).$$

To specify the estimates and decreasing properties of the stable part we use the following essential assumption.

Assumption 1: We assume that the graph norm of $\mathcal{D}(A(t))$ is uniformly-in-time equivalent to the norm of a fixed Banach space D.

The estimates and stabilisation properties proved in [42] can be adapted directly to this framework, replacing the graph norm of the operator with the D-norm. Indeed the assumption that the domain of $A(t)$ is constant in [40, 42] is used to ensure the existence of an evolution operator. The asymptotic behaviour of $U_s(t_1, t_2)$ in E_0, for $\varepsilon > 0$ small enough,

$$\|U_s(t_1, t_2)\|_{\mathcal{L}(E_0)} \leq k_1(\varepsilon)e^{-(\omega+\varepsilon)(t_1-t_2)}, \;\; t_1 > t_2, \tag{4.3.6}$$

is proved in [40, Proposition 2.3]. The estimates in higher norm follow from (4.3.6) using $U_s \in \mathcal{C}(\dot{\Delta}; \mathcal{L}(E_0, D))$ when the graph norm of $\mathcal{D}(A(t))$ is equivalent to the D-norm; see also Remark 4.6.1. We obtain, for $\varepsilon > 0$ small enough,

$$\|U_s(t_1, t_2)\|_{\mathcal{L}(E_0,D)} \le k_1(\varepsilon)(t-s)^{-1} e^{-(\omega+\varepsilon)(t_1-t_2)}, \tag{4.3.7}$$

using the property (U_4) of the parabolic evolution operator.

Consider now the parabolic evolution equation

$$\begin{cases} y'(t) = A(t)y(t) + B(t)f(t), \;\; t > 0, \\ y(0) = x, \end{cases} \tag{4.3.8}$$

where $B \in \mathcal{C}^{\rho}_{\sharp}([0,T]; \mathcal{L}(Y, E_0))$ is the control operator with Y a Hilbert space. Let $f \in \mathcal{C}^{\rho}([0,+\infty); Y)$. Using [4, Theorem 2.5.1, Section IV], for every $x \in E_0$ the Cauchy problem (4.3.8) admits a unique classical solution

$$y \in \mathcal{C}([0;+\infty); E_0) \cap \mathcal{C}^1((0,+\infty); E_0),$$

with $y(0) = x$ and $y(t) \in \mathcal{D}(A(t))$ for all $t > 0$. Using the uniform-in-time equivalence of the graph norm of $\mathcal{D}(A(t))$ and the D-norm, the fact that $y' \in \mathcal{C}((0,\infty), E_0)$, and the equation $y'(t) = A(t)y(t) + B(t)f(t)$, we deduce that

$$y \in \mathcal{C}([0;+\infty); E_0) \cap \mathcal{C}((0,+\infty); D) \cap \mathcal{C}^1((0,+\infty); E_0).$$

Following [40] we introduce the spaces, and related norms, of exponentially decaying functions:

$$\begin{aligned}
\mathcal{C}_{\omega_1}([0;+\infty); D) &= \left\{ \mathbf{u} \in \mathcal{C}([0,+\infty); D) \Big| \sup_{t>0} \left\| \mathbf{u}(t) e^{\omega_1 t} \right\|_D < +\infty \right\}, \\
\|\mathbf{u}\|_{\mathcal{C}_{\omega_1}([0;+\infty);D)} &= \sup_{t>0} \left\| \mathbf{u}(t) e^{\omega_1 t} \right\|_D, \\
\mathcal{C}^{\rho_1}_{\omega_1}([0,+\infty); D) &= \left\{ \mathbf{u} \in \mathcal{C}_{\omega_1}([0,+\infty); D) \Big| \sup_{0<s<t} \left\| \mathbf{u}(t) e^{\omega_1 t} - \mathbf{u}(s) e^{\omega_1 s} \right\|_D (t-s)^{-\rho_1} < +\infty \right\}, \\
\|\mathbf{u}\|_{\mathcal{C}^{\rho_1}_{\omega_1}([0,+\infty);D)} &= \sup_{t>0} \left\| \mathbf{u}(t) e^{\omega_1 t} \right\|_D + \sup_{0<s<t} \left\| \mathbf{u}(t) e^{\omega_1 t} - \mathbf{u}(s) e^{\omega_1 s} \right\|_D (t-s)^{-\rho_1},
\end{aligned}$$

with $\omega_1 \ge 0$ and $0 < \rho_1 < 1$. If $\omega_1 = 0$, $\mathcal{C}_0([0;+\infty); D)$ consists of bounded functions, and $\mathcal{C}^{\rho_1}_0([0,+\infty); D)$ of uniformly Hölder continuous functions.

The following theorem can be deduced from [42, Theorem 3.1] in our framework.

Theorem 4.3.4. Suppose that the following implication holds:

$$\forall \lambda \in \mathbb{C},\ |\lambda| > e^{-\omega T},\ (\lambda - V^*(0))x^* = 0,\ B^*(\cdot)U^*(T,\cdot)x^* \equiv 0 \text{ on } [0,T] \;\Rightarrow x^* = 0. \tag{4.3.9}$$

Then there exists $f \in \mathcal{C}^{\rho}_{\omega}([0;+\infty); Y)$ such that the classical solution y to (4.3.8) belongs to $\mathcal{C}_{\omega}([a,+\infty); D)$ for all $a > 0$.

Remark 4.3.2. A sufficient condition to obtain (4.3.9), that we use later to stabilise the linear fluid–structure system, is the following implication:

$$-q' = A^*(t)q \text{ in } [0,T],\ \ B^*(\cdot)q \equiv 0 \text{ on } [0,T] \ \Rightarrow\ q \equiv 0 \text{ in } [0,T]. \tag{4.3.10}$$

Indeed $q(t) = U^*(T,t)x^*$ is a solution to $-q' = A^*(t)q$ in $[0,T]$.

4.4 The perturbed Oseen system

In this section we study the properties of a perturbed Oseen system. The objective is to rewrite the fluid equation in (4.2.9) without the pressure. This can be done using the Leray projector. To remove the pressure term in the beam equation we also need to express the pressure in terms of the velocity of the fluid.

For $t \geq 0$ and $\lambda_0 > 0$ consider the system

$$\begin{aligned} &\lambda_0\mathbf{u} - \nu\Delta\mathbf{u} + (\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{u}}_\pi(t) + C_\pi(t)\mathbf{u}_z + \nabla p = \mathbf{f} \ \text{ in } \Omega_{\pi,0},\\ &\operatorname{div}\mathbf{u} = w \ \text{ in } \Omega_{\pi,0},\ \ \mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_{\pi,0},\ \ \mathbf{u} = \mathbf{u}_c \ \text{ on } \Gamma_i,\\ &u_2 = 0 \ \text{ and } p = 0 \text{ on } \Gamma_o,\ \ \mathbf{u} = 0 \ \text{ on } \Gamma_b. \end{aligned} \tag{4.4.1}$$

The perturbation terms, with respect to the Stokes system, are denoted by

$$\begin{aligned} &D_\pi(t)\mathbf{u} = (\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{u}}_\pi(t) + C_\pi(t)\mathbf{u}_z\\ &D^a_\pi(t)\mathbf{u} = -(\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{u} + (\nabla\overline{\mathbf{u}}_\pi(t))^T\mathbf{u} - \mathbf{u}\operatorname{div}(\overline{\mathbf{u}}_\pi(t)) - C_{\pi,z}(t)\mathbf{u} + C_\pi(t)\mathbf{u}_z, \end{aligned}$$

where $D^a_\pi(t)$ is the perturbation term in the adjoint equation of (4.4.1) (see Theorem 4.4.3).

Lemma 4.4.1. Let $\mathbf{u}$ be in $\mathbf{H}^1(\Omega_{\pi,0})$ and $\mathbf{v}$ be in $\mathbf{H}^{1+\varepsilon}(\Omega_{\pi,0})$ with $\varepsilon > 0$. The following estimates hold

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{\mathbf{L}^2(\Omega_{\pi,0})} + \|(\mathbf{v}\cdot\nabla)\mathbf{u}\|_{\mathbf{L}^2(\Omega_{\pi,0})} \leq C\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega_{\pi,0})}\,\|\mathbf{v}\|_{\mathbf{H}^{1+\varepsilon}(\Omega_{\pi,0})}\,.$$

Proof. The term $(\mathbf{v}\cdot\nabla\mathbf{u})$ can be estimated directly as an $\mathbf{L}^\infty\times\mathbf{L}^2$ product. For $(\mathbf{u}\cdot\nabla)\mathbf{v}$ we use [30, Proposition B.1] which implies the estimate

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{\mathbf{L}^2(\Omega_{\pi,0})} \leq C\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega_{\pi,0})}\,\|\nabla\mathbf{v}\|_{\mathbf{H}^\varepsilon(\Omega_{\pi,0})^2} \leq C\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega_{\pi,0})}\,\|\mathbf{v}\|_{\mathbf{H}^{1+\varepsilon}(\Omega_{\pi,0})}\,.$$

□

We fix $\lambda_0 > 0$ such that the following inequality holds

$$\lambda_0\int_{\Omega_{\pi,0}}|\mathbf{u}|^2 + \nu\int_{\Omega_{\pi,0}}|\nabla\mathbf{u}|^2 + \int_{\Omega_{\pi,0}}D_\pi(t)\mathbf{u}\cdot\mathbf{u} + \int_{\Omega_{\pi,0}}D^a_\pi(t)\mathbf{u}\cdot\mathbf{u} \geq \frac{\nu}{2}\,\|\mathbf{u}\|^2_V\,, \tag{4.4.2}$$

for all $t \geq 0$ and all $\mathbf{u} \in V$. We consider non-homogeneous divergences belonging to

$$H^1_{\text{lift}}(\Omega_{\pi,0}) := \{\text{div } \mathbf{u} \mid \mathbf{u} \in \mathbf{H}^2(\Omega_{\pi,0}) \cap \mathbf{H}^1_0(\Omega_{\pi,0})\}.$$

Taking this space simplifies the lifting of the non-homogeneous divergence, which becomes a translation. The term $A_3\hat{\eta}_1$ in the fluid–structure system (4.2.9) belongs to this space.

Theorem 4.4.1. For all $(\mathbf{f}, w, \mathbf{g}, \mathbf{u}_c) \in \mathbf{L}^2(\Omega_{\pi,0}) \times H^1_{\text{lift}}(\Omega_{\pi,0}) \times \mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0}) \times \mathbf{H}^{3/2}_0(\Gamma_i)$, (4.4.1) admits a unique solution $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ which satisfies

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbf{H}^2(\Omega_{\pi,0})} + \|p\|_{H^1(\Omega_{\pi,0})} \\ &\qquad \leq C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\pi,0})} + \|w\|_{H^1(\Omega_{\pi,0})} + \|\mathbf{g}\|_{\mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0})} + \|\mathbf{u}_c\|_{\mathbf{H}^{3/2}_0(\Gamma_i)}). \end{aligned} \tag{4.4.3}$$

Proof. The theorem can be proved in three steps:

- *Step 1:* We lift the non-homogeneous terms in (4.4.1).
- *Step 2:* We study a variational formulation associated to (4.4.1) and we prove the existence of weak solution.
- *Step 3:* We use the strong regularity result proved in Chapter 2 for the Stokes system with pressure boundary conditions and a bootstrap argument to conclude.

Step 1: Consider the system

$$\begin{aligned} &\text{div } \mathbf{\Phi} = 0 \text{ in } \Omega_{\pi,0}, \\ &\mathbf{\Phi} = \mathbf{g} \text{ on } \Gamma_{\pi,0},\ \mathbf{\Phi} = 0 \text{ on } \Gamma_i \cup \Gamma_b,\ \Phi_2 = 0 \text{ on } \Gamma_o. \end{aligned} \tag{4.4.4}$$

Let us perform the change of variables

$$\mathcal{T}_{\eta^0_{\pi,1}} : \begin{cases} \Omega_{\eta_{\pi,0}} \longrightarrow \Omega \\ (x,y) \mapsto (x,z) = \left(x, \frac{1}{1+\eta^0_{\pi,1}(x)} y\right). \end{cases}$$

With $\hat{\mathbf{\Phi}}(x,z) = \mathbf{\Phi}(\mathcal{T}^{-1}_{\eta^0_{\pi,1}}(x,z))$, system (4.4.4) becomes

$$\begin{aligned} &\text{div } \hat{\mathbf{\Phi}} = \text{div}(\mathbf{w}(\hat{\mathbf{\Phi}})) \text{ in } \Omega, \\ &\hat{\mathbf{\Phi}} = \hat{\mathbf{g}} \text{ on } \Gamma_s,\ \hat{\mathbf{\Phi}} = 0 \text{ on } \Gamma_i \cup \Gamma_b,\ \hat{\Phi}_2 = 0 \text{ on } \Gamma_o, \end{aligned} \tag{4.4.5}$$

with $\hat{\mathbf{g}} = \mathbf{g} \circ \mathcal{T}^{-1}_{\eta^0_{\pi,1}}$ and $\mathbf{w}(\hat{\mathbf{\Phi}}) = -\eta^0_{\pi,1}\hat{\Phi}_1\mathbf{e}_1 + z\eta^0_{\pi,1,x}\hat{\Phi}_1\mathbf{e}_2$. Then we consider another change of unknown to lift the non-homogeneous divergence $\mathbf{v} = \hat{\mathbf{\Phi}} - \mathbf{w}(\hat{\mathbf{\Phi}})$. The previous relation can be written

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 + \eta^0_{\pi,1} & 0 \\ -z\eta^0_{\pi,1,x} & 1 \end{pmatrix}}_{=:A(\eta^0_{\pi,1})} \begin{pmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{pmatrix}.$$

The matrix $A(\eta^0_{\pi,1})$ is invertible and, noticing that $\mathbf{w}(\hat{\boldsymbol{\Phi}}) = 0$ on $\partial\Omega$, the system (4.4.5) is equivalent to

$$\begin{aligned} &\text{div } \mathbf{v} = 0 \text{ in } \Omega, \\ &\mathbf{v} = \hat{\mathbf{g}} \text{ on } \Gamma_s,\ \mathbf{v} = 0 \text{ on } \Gamma_i \cup \Gamma_b,\ v_2 = 0 \text{ on } \Gamma_o. \end{aligned} \tag{4.4.6}$$

In order to solve (4.4.6) we consider a Stokes system with Dirichlet boundary conditions on an extended domain. Set

$$\overline{\mathbf{g}} : \begin{cases} \overline{\mathbf{g}} = \hat{\mathbf{g}} \text{ on } (0, L) \times \{1\}, \\ \overline{\mathbf{g}}(x, 1) = -\hat{\mathbf{g}}(2L - x, 1) \text{ for } x \in (L, 2L). \end{cases}$$

Thanks to the properties of the space $H^{3/2}_{00}(\Gamma_s)$, the function $\overline{\mathbf{g}}$ is in $\mathcal{H}^{3/2}_{00}((0, 2L) \times \{1\})$. Moreover the function $\overline{\mathbf{g}}$ has a zero average. We set $\Omega_e = (0, 2L) \times (0, 1)$, $\Gamma_{s,e} = (0, 2L) \times \{1\}$, $\Gamma_{b,e} = (0, 2L) \times \{0\}$ and $\Gamma_{o,e} = \{2L\} \times (0, 1)$. Consider the Stokes system

$$\begin{aligned} &-\nu\Delta\boldsymbol{\phi} + \nabla q = 0,\ \ \text{div } \boldsymbol{\phi} = 0 \ \text{ in } \Omega_e, \\ &\boldsymbol{\phi} = \overline{\mathbf{g}} \ \text{ on } \Gamma_{s,e},\ \boldsymbol{\phi} = 0 \ \text{ on } \partial\Omega_e \setminus \Gamma_{s,e}. \end{aligned} \tag{4.4.7}$$

This system admits a unique solution $(\boldsymbol{\phi}, q) \in \mathbf{H}^2(\Omega_e) \times H^1(\Omega_e)$ (see for example [47]). We introduce the function

$$\boldsymbol{\phi}_s(x, z) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{\phi}(2L - x, z) \text{ for all } (x, z) \in \Omega_e.$$

The function $\boldsymbol{\phi}_s \in \mathbf{H}^2(\Omega_e)$ still satisfies

$$\begin{aligned} &\text{div } \boldsymbol{\phi}_s = 0 \text{ in } \Omega_e, \\ &\boldsymbol{\phi}_s = \overline{\mathbf{g}} \text{ on } \Gamma_{s,e},\ \boldsymbol{\phi}_s = 0 \text{ on } \partial\Omega_e \setminus \Gamma_{s,e}. \end{aligned}$$

Finally set $\overline{\boldsymbol{\phi}} = \frac{\boldsymbol{\phi}+\boldsymbol{\phi}_s}{2}$. Noticing that $\overline{\phi}_2(L, z) = 0$ for all $z \in (0, 1)$ the restriction to Ω of $\overline{\boldsymbol{\phi}}$ is solution to (4.4.6). Hence, coming back to the initial system, we have proved the existence of $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathbf{g}}_{\pi,0} \in \mathbf{H}^2(\Omega_{\pi,0})$ solution to (4.4.4).

To lift the non-homogeneous Dirichlet boundary condition on the inflow we want to use the regularity results of [47] for the Stokes system in a dihedron. The first step is to 'compensate' the non zero average of $\mathbf{u}_c \cdot \mathbf{n}$. Consider a function $\varphi \in \mathcal{C}^\infty_0(\Gamma_{\pi,0})$ such that $\int_{\Gamma_{\pi,0}} \varphi \neq 0$. Let $\mathbf{u}_h \in \mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0})$ be the function defined by

$$\mathbf{u}_h(x, y) = -\frac{\varphi(x)}{\int_{\Gamma_{\pi,0}} \varphi} \left(\int_{\Gamma_i} \mathbf{u}_c \cdot \mathbf{n} \right) \mathbf{e}_2,\ \ \forall (x, y) \in \Gamma_{\pi,0}.$$

Consider the Stokes system

$$\begin{aligned} &-\nu\Delta\boldsymbol{\phi} + \nabla q = 0,\ \ \text{div } \boldsymbol{\phi} = 0 \ \text{ in } \Omega_{\pi,0}, \\ &\boldsymbol{\phi} = \mathbf{u}_h \ \text{ on } \Gamma_{\pi,0},\ \boldsymbol{\phi} = \mathbf{u}_c \ \text{ on } \Gamma_i,\ \boldsymbol{\phi} = 0 \text{ on } \Gamma_b \cup \Gamma_o. \end{aligned}$$

Using [47] we obtain a solution $(\boldsymbol{\phi}, q) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$. We then consider $\boldsymbol{\Phi}_i^{\mathbf{u}_c} = \boldsymbol{\phi} - \boldsymbol{\Phi}_{\pi,0}^{\mathbf{u}_h}$, where $\boldsymbol{\Phi}_{\pi,0}^{\mathbf{u}_h}$ is a solution to (4.4.4) with $\boldsymbol{\Phi} = \mathbf{u}_h$ on $\Gamma_{\pi,0}$, to obtain the desired lifting for $\mathbf{u}_c$.

The non-homogeneous divergence condition in (4.4.1) can be lifted directly as we choose w under the form div $\mathbf{w}$ with $\mathbf{w} \in \mathbf{H}^2(\Omega_{\pi,0}) \cap \mathbf{H}_0^1(\Omega_{\pi,0})$. If we set $\overline{\mathbf{u}} = \mathbf{u} - \boldsymbol{\Phi}_{\pi,0}^{\mathbf{g}} - \boldsymbol{\Phi}_i^{\mathbf{u}_c} - \mathbf{w}$, the pair $(\mathbf{u}, p)$ satisfies (4.4.1) if and only if the pair $(\overline{\mathbf{u}}, p)$ is solution to

$$
\begin{aligned}
&\lambda_0 \overline{\mathbf{u}} - \nu \Delta \overline{\mathbf{u}} + (\overline{\mathbf{u}}_\pi(t) \cdot \nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}}_\pi(t) + C_\pi(t)\overline{\mathbf{u}}_z + \nabla p = \mathbf{F} \ \text{ in } \Omega_{\pi,0}, \\
&\text{div } \overline{\mathbf{u}} = 0 \ \text{ in } \Omega_{\pi,0}, \ \ \mathbf{u} = 0 \ \text{ on } \partial\Omega_{\pi,0} \setminus \Gamma_o, \ \ u_2 = 0 \ \text{ and } p = 0, \ \text{ on } \Gamma_o,
\end{aligned}
\tag{4.4.8}
$$

with $\mathbf{F} = \mathbf{f} - \lambda_0(\boldsymbol{\Phi}_{\pi,0}^{\mathbf{g}} + \boldsymbol{\Phi}_i^{\mathbf{u}_c} + \mathbf{w}) - D_\pi(t)(\boldsymbol{\Phi}_{\pi,0}^{\mathbf{g}} + \boldsymbol{\Phi}_i^{\mathbf{u}_c} + \mathbf{w})$.

Step 2: Following the study of the Stokes system with pressure boundary condition in Chapter 2 we introduce the variational formulation:

$$
\begin{aligned}
&\text{Find } \overline{\mathbf{u}} \in V \text{ such that} \\
&\lambda_0 \int_{\Omega_{\pi,0}} \overline{\mathbf{u}} \cdot \boldsymbol{\varphi} + \nu \int_{\Omega_{\pi,0}} \nabla \overline{\mathbf{u}} : \nabla \boldsymbol{\varphi} + \int_{\Omega_{\pi,0}} D_\pi(t)\overline{\mathbf{u}} \cdot \boldsymbol{\varphi} = \int_{\Omega_{\pi,0}} \mathbf{F} \cdot \boldsymbol{\varphi}, \ \text{ for all } \boldsymbol{\varphi} \in V.
\end{aligned}
\tag{4.4.9}
$$

Using the Lax-Milgram theorem, and (4.4.2) to ensure the coercivity of the bilinear form, the variational formulation (4.4.9) admits a unique solution $\overline{\mathbf{u}} \in V$. Moreover, proceeding as in Theorem 2.5.3, there exists a pressure $\mathcal{P} \in L^2(\Omega_{\pi,0})$, unique up to an additive constant, such that $\lambda_0 \overline{\mathbf{u}} - \nu \Delta \overline{\mathbf{u}} + D_\pi(t)\overline{\mathbf{u}} + \nabla \mathcal{P} = \mathbf{F}$ in $\mathbf{H}^{-1}(\Omega_{\pi,0})$.

Step 3: Defining the linear form on V

$$
b(\boldsymbol{\varphi}) = \int_{\Omega_{\pi,0}} \mathbf{F} \cdot \boldsymbol{\varphi} - \lambda_0 \int_{\Omega_{\pi,0}} \overline{\mathbf{u}} \cdot \boldsymbol{\varphi} - \int_{\Omega_{\pi,0}} D_\pi(t)\overline{\mathbf{u}} \cdot \boldsymbol{\varphi},
$$

the function $\overline{\mathbf{u}}$ satisfies

$$
\nu \int_{\Omega_{\pi,0}} \nabla \overline{\mathbf{u}} : \nabla \boldsymbol{\varphi} = b(\varphi), \ \text{ for all } \boldsymbol{\varphi} \in V.
$$

Hence, using Theorem 2.5.4, we recover that $\overline{\mathbf{u}} \in \mathbf{H}^2(\Omega_{\pi,0})$ and its associated pressure $\mathcal{P}$ is in $H^1(\Omega_{\pi,0})$ and is constant on Γ_o. We then conclude as in the steps 4–5 in Theorem 2.5.4 by taking the only pressure p in the class of $\mathcal{P}$ which satisfies $p = 0$ on Γ_o. $\square$

Remark 4.4.1. The lifting of the non-homogeneous boundary condition on $\Gamma_{\pi,0}$ that we build in the first step of the proof of Theorem 4.4.1 provides another proof of Theorem 2.5.1. Indeed if $\boldsymbol{\Phi}_1$ is solution to (4.4.4) and $\boldsymbol{\Phi}_2$ is solution to (4.4.4) where the boundary conditions on Γ_i and Γ_o have been switched, we can find a convex combination of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ that satisfies (4.4.4) and with second component that vanishes on $\Gamma_i \cup \Gamma_o$.

4.4.1 Expression of the pressure

In order to express the pressure several approaches are possible. The method used in Chapter 2, which consists in applying $(I - \Pi)$ on the fluid equations and use the elliptic equations defining Π, can be applied here as the perturbed Oseen system possesses $\mathbf{H}^2$-solutions.

Another technique to express the pressure, introduce in [23], consists in multiplying the fluid equation by test function χ_ξ defined as follows. For $\xi \in L^2(\Omega_{\pi,0})$ let $\chi_\xi \in H^2(\Omega_{\pi,0})$ be the solution to

$$\begin{aligned} &\Delta\chi_\xi = \xi \ \text{ in } \Omega_{\pi,0}, \\ &\frac{\partial \chi_\xi}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d, \\ &\chi_\xi = 0 \ \text{ on } \Gamma_o. \end{aligned} \tag{4.4.10}$$

This method was introduced by the authors to compensate the lack of regularity induced by the mixed boundary conditions. In our case the Stokes/Oseen system still possesses a solution in $\mathbf{H}^2(\Omega_{\pi,0})$ thanks to the symmetry argument in Chapter 2. The second method is however more ‘compact’ in that case and directly provides the expected regularity result, so we use this approach for our problem. Multiplying (4.4.1) by $\nabla\chi_\xi$ and integrating over $\Omega_{\pi,0}$ we obtain

$$\lambda_0 \int_{\Omega_{\pi,0}} \mathbf{u}\cdot\nabla\chi_\xi + \int_{\Omega_{\pi,0}} D_\pi(t)\mathbf{u}\cdot\nabla\chi_\xi - \nu\int_{\Omega_{\pi,0}} \Delta\mathbf{u}\cdot\nabla\chi_\xi + \int_{\Omega_{\pi,0}} \nabla p\cdot\nabla\chi_\xi = \int_{\Omega_{\pi,0}} \mathbf{f}\cdot\nabla\chi_\xi.$$

Using integrations by parts

$$\begin{aligned} \int_{\Omega_{\pi,0}} p\,\xi = &- \int_{\Omega_{\pi,0}} \mathbf{f}\cdot\nabla\chi_\xi + \nu\int_{\Omega_{\pi,0}} \nabla\mathbf{u} : \nabla^2\chi_\xi - \nu\int_{\partial\Omega_{\pi,0}} (\nabla\mathbf{u}\,\nabla\chi_\xi)\cdot\mathbf{n} \\ &+ \int_{\Omega_{\pi,0}} D_\pi(t)\mathbf{u}\cdot\nabla\chi_\xi - \lambda_0\int_{\Omega_{\pi,0}} \chi_\xi \operatorname{div}\,\mathbf{u} + \lambda_0\int_{\Gamma_{\pi,0}} \chi_\xi\,\mathbf{u}\cdot\mathbf{n} + \lambda_0\int_{\Gamma_i} \chi_\xi\,\mathbf{u}\cdot\mathbf{n}. \end{aligned} \tag{4.4.11}$$

The right-hand side of the equation is a linear form on ξ. We introduce the following operators:

- For $0 < \varepsilon_0 < 1/2$, $N_v \in \mathcal{L}(\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\pi,0}), L^2(\Omega_{\pi,0}))$ is defined by $N_v(\mathbf{u}) = q_1$ where q_1 is given by the Riesz representation theorem and the identity, for all $\xi \in L^2(\Omega_{\pi,0})$,

$$\int_{\Omega_{\pi,0}} q_1\,\xi = \nu\int_{\Omega_{\pi,0}} \nabla\mathbf{u} : \nabla^2\chi_\xi - \nu\int_{\partial\Omega_{\pi,0}} (\nabla\mathbf{u}\,\nabla\chi_\xi)\cdot\mathbf{n} + \int_{\Omega_{\pi,0}} D_\pi(t)\mathbf{u}\cdot\nabla\chi_\xi. \tag{4.4.12}$$

- $N_s \in \mathcal{L}(\mathbf{L}^2(\Gamma_{\pi,0}), L^2(\Omega_{\pi,0}))$ is defined by $N_s(\mathbf{g}) = q_2$ with

$$
\begin{aligned}
&\Delta q_2 = 0 \ \text{ in } \Omega_{\pi,0},\\
&\frac{\partial q_2}{\partial \mathbf{n}} = \mathbf{g}\cdot\mathbf{n} \ \text{ on } \Gamma_{\pi,0},\\
&\frac{\partial q_2}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_i \cup \Gamma_b,\\
&q_2 = 0 \ \text{ on } \Gamma_o.
\end{aligned}
\tag{4.4.13}
$$

- $N_i \in \mathcal{L}(\mathbf{L}^2(\Gamma_i), L^2(\Omega_{\pi,0}))$ is defined by $N_s(\mathbf{g}) = q_2$ with

$$
\begin{aligned}
&\Delta q_3 = 0 \ \text{ in } \Omega_{\pi,0},\\
&\frac{\partial q_3}{\partial \mathbf{n}} = \mathbf{u}\cdot\mathbf{n} \ \text{ on } \Gamma_i,\\
&\frac{\partial q_3}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_{\pi,0} \cup \Gamma_b,\\
&q_3 = 0 \ \text{ on } \Gamma_o.
\end{aligned}
\tag{4.4.14}
$$

- $N_{\mathrm{div}} \in \mathcal{L}(L^2(\Omega_{\pi,0}), H^1(\Omega_{\pi,0}))$ is defined by $N_{\mathrm{div}}(w) = q_4$ with

$$
\begin{aligned}
&\Delta q_4 = w \ \text{ in } \Omega_{\pi,0},\\
&\frac{\partial q_4}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_d,\\
&q_4 = 0 \ \text{ on } \Gamma_o.
\end{aligned}
\tag{4.4.15}
$$

- $N_p \in \mathcal{L}(\mathbf{L}^2(\Omega_{\pi,0}), H^1_{\Gamma_o}(\Omega_{\pi,0}))$ is defined by $N_p(\mathbf{f}) = q_5$ with $(I - \Pi)\mathbf{f} = \nabla q_5$.

Lemma 4.4.2. Let $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ be a solution to (4.4.1). Then the pressure p is determined by

$$
p = -\lambda_0 N_{\mathrm{div}}(w) - \lambda_0 N_i(\mathbf{u}_c) - \lambda_0 N_s(\mathbf{g}) + N_v(\mathbf{u}) + N_p(\mathbf{f}).
$$

Proof. Using that $\nabla\chi_\xi \in \nabla H^1_{\Gamma_o}(\Omega_{\pi,0})$ and Green formula, the following identities hold

$$
-\int_{\Omega_{\pi,0}} \mathbf{f}\cdot\nabla\chi_\xi = -\int_{\Omega_{\pi,0}} (I-\Pi)\mathbf{f}\cdot\nabla\chi_\xi = -\int_{\Omega_{\pi,0}} \nabla N_p(\mathbf{f})\cdot\nabla\chi_\xi = \int_{\Omega_{\pi,0}} N_p(\mathbf{f})\,\xi,
$$

$$
\nu\int_{\Omega_{\pi,0}} \nabla\mathbf{u} : \nabla^2\chi_\xi - \nu\int_{\partial\Omega_{\pi,0}} (\nabla\mathbf{u}\,\nabla\chi_\xi)\cdot\mathbf{n} + \int_{\Omega_{\pi,0}} D_\pi(t)\mathbf{u}\cdot\nabla\chi_\xi = \int_{\Omega_{\pi,0}} N_v(\mathbf{u})\,\xi,
$$

$$
-\lambda_0\int_{\Omega_{\pi,0}} \chi_\xi \operatorname{div}\mathbf{u} = -\lambda_0\int_{\Omega_{\pi,0}} \chi_\xi\, h = -\lambda_0\int_{\Omega_{\pi,0}} \chi_\xi\,\Delta N_{\mathrm{div}}(h) = -\lambda_0\int_{\Omega_{\pi,0}} N_{\mathrm{div}}(h)\,\xi,
$$

$$
\lambda_0\int_{\Gamma_{\pi,0}} \chi_\xi\,\mathbf{u}\cdot\mathbf{n} = \lambda_0\int_{\Gamma_{\pi,0}} \chi_\xi\frac{\partial N_s(\mathbf{g})}{\partial\mathbf{n}} = \lambda_0\int_{\Omega_{\pi,0}} \nabla\chi_\xi\cdot\nabla N_s(\mathbf{g}) = -\lambda_0\int_{\Omega_{\pi,0}} N_s(\mathbf{g})\,\xi,
$$

$$\lambda_0 \int_{\Gamma_i} \chi_\xi \, \mathbf{u}\cdot\mathbf{n} = \lambda_0 \int_{\Gamma_i} \chi_\xi \frac{\partial N_i(\mathbf{u}_c)}{\partial \mathbf{n}} = \lambda_0 \int_{\Omega_{\pi,0}} \nabla\chi_\xi \cdot \nabla N_i(\mathbf{u}_c) = -\lambda_0 \int_{\Omega_{\pi,0}} N_i(\mathbf{u}_c)\, \xi.$$

Hence (4.4.11) becomes

$$\int_{\Omega_{\pi,0}} p\,\xi = \int_{\Omega_{\pi,0}} \left[-\lambda_0 N_{\mathrm{div}}(w) - \lambda_0 N_s(\mathbf{g}) - \lambda_0 N_i(\mathbf{u}_c) + N_v(\mathbf{u}) + N_p(F)\right] \xi,$$

which concludes the proof. □

We specify some regularity results for the operators N_s, N_i and N_{div} which are useful in the study of the fluid structure system.

Lemma 4.4.3. The operator N_s and N_i can be extended as follows:

- $N_s \in \mathcal{L}(\mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0}), H^3(\Omega_{\pi,0}))$, $N_i \in \mathcal{L}(\mathbf{H}^{3/2}_0(\Gamma_i), H^3(\Omega_{\pi,0}))$.
- $N_s \in \mathcal{L}(\mathcal{H}^{3/2}(\Gamma_{\pi,0})', L^2(\Omega_{\pi,0}))$, $N_i \in \mathcal{L}(\mathbf{H}^{-3/2}(\Gamma_i), L^2(\Omega_{\pi,0}))$.
- $N_s \in \mathcal{L}(\mathcal{H}^{1/2}(\Gamma_{\pi,0})', H^1(\Omega_{\pi,0}))$, $N_i \in \mathcal{L}(\mathbf{H}^{-1/2}(\Gamma_i), H^1(\Omega_{\pi,0}))$.

The operator N_{div} can be extended as an operator in $\mathcal{L}(H^1_{\Gamma_o}(\Omega_{\pi,0}), H^3(\Omega_{\pi,0}))$.

Proof. See Lemma 2.5.27 for the first regularity result on N_s, N_i and the regularity result on N_{div}. The extension of N_s (respectively N_i) to $\mathcal{H}^{-3/2}(\Gamma_{\pi,0})$ (respectively $\mathbf{H}^{-3/2}(\Gamma_i)$) can be done by duality. Finally the last statements on N_s and N_i are interpolated versions of the previous results. □

4.4.2 The perturbed Oseen operator

Let $(A_2(t), \mathcal{D}(A_2(t)))_{t\geq 0}$ be the operator in $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$ defined by $\mathcal{D}(A_2(t)) = V$ and $A_2(t)\mathbf{u} = -\Pi D_\pi(t)\mathbf{u}$ for all $\mathbf{u} \in V$. Introduce the perturbed Oseen operator $(A(t), \mathcal{D}(A(t)))_{t\geq 0}$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$ defined by $\mathcal{D}(A(t)) = V \cap \mathbf{H}^2(\Omega_{\pi,0})$ and $A(t) = A_s + A_2(t)$ where A_s is the Stokes operator on $\Omega_{\pi,0}$ defined as in (4.2.4).

Theorem 4.4.2. The family of closed operators $(A(t), \mathcal{D}(A(t))_{t\geq 0}$ satisfies the assumptions (A_1)–(A_3) and there exists a unique parabolic evolution operator with regularity subspace $V \cap \mathbf{H}^2(\Omega_{\pi,0})$.

Proof. First notice that $\mathcal{D}(A(t)) = V \cap \mathbf{H}^2(\Omega_{\pi,0})$ does not depend on t. As in Theorem 2.5.5 it can be proved that the unbounded operator $(A_s, \mathcal{D}(A_s))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$ and that $\mathcal{D}(A_s^{1/2}) = V$. Hence there exists $\lambda \in \mathbb{R}$ and $M > 0$ such that $A_s \in \mathfrak{A}_{M,\lambda}(\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}))$. The family of unbounded operator $(A_2(t), \mathcal{D}(A_2(t)))_{t\geq 0}$ is in $\mathcal{L}(V, \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}))$ and $\|A_2(t)\|_{\mathcal{L}(V,\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}))} \leq \overline{C}$ where the constant $\overline{C}$ does not depend on t. Indeed for all $\mathbf{u} \in V$, using Lemma 4.4.1, the following estimates hold

$$\|\Pi(\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{u}\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})} + \|\Pi(\mathbf{u}\cdot\nabla)\overline{\mathbf{u}}_\pi(t)\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})} \leq C\,\|\overline{\mathbf{u}}_\pi\|_{\mathcal{C}^\rho([0,T];\mathbf{H}^2(\Omega_{\pi,0}))}\,\|\mathbf{u}\|_V\,.$$

Let us recall that

$$C_\pi(t)\mathbf{u}_z = \frac{z\tilde{\eta}_{\pi,t}}{1+\tilde{\eta}_\pi}\mathbf{u}_z = \frac{z\eta_{\pi,t}}{1+\eta_\pi}\mathbf{u}_z,$$

and define $\mu_\pi = \left\|(1+\eta_\pi)^{-1}\right\|_{L^\infty(\Sigma^s_T)}$. Then

$$\|\Pi C_\pi(t)\mathbf{u}_z\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})} \leq C\mu_\pi\,\|\eta_\pi\|_{\mathcal{C}^\rho([0,T];H^4(\Gamma_s))}\,\|\mathbf{u}\|_V\,.$$

The previous estimates also prove that $A(\cdot) \in \mathcal{C}^\rho([0,T];\mathcal{L}(\mathcal{D}(A_s),\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})))$. Finally using Theorem 4.3.3 (with $\kappa = 1/2$) we obtain that $A_s + A_2 : [0,T] \to \mathfrak{A}_{M',\lambda'}(\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}))$ for $\lambda' \in \mathbb{R}$ and $M' > 0$ independent of t. $\square$

Let $L_{\Gamma_o} \in \mathcal{L}(H^{1/2}(\Gamma_o), H^1(\Omega_{\pi,0}))$ be a lifting operator and set $L_o(\mathbf{u}) = L_{\Gamma_o}((\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{n})u_1)$.

Theorem 4.4.3. The adjoint of $(A(t),\mathcal{D}(A(t)))_{t\geq 0}$ in $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$ is defined by

$$\begin{aligned}&\mathcal{D}(A^*(t)) = V \cap \mathbf{H}^2(\Omega_{\pi,0}) \text{ and, for } \mathbf{u} \in \mathcal{D}(A^*(t)),\\ &A^*(t)\mathbf{u} = \Pi(\nu\Delta\mathbf{u} - D^a_\pi(t)\mathbf{u} - \nabla L_o(\mathbf{u})).\end{aligned}$$

Proof. The proof relies on integration by parts. The computations provided in Lemma 4.5.4 for the coupled system directly imply that the adjoint equation of (4.4.1) (with $w = 0$, $\mathbf{g} = 0$ and $\mathbf{u}_c = 0$) is

$$\begin{aligned}&\lambda_0\mathbf{v} - \nu\Delta\mathbf{v} + D^a_\pi(t)\mathbf{v} + \nabla q = \Phi \ \text{ in } \Omega_{\pi,0},\\ &\operatorname{div}\mathbf{v} = 0 \ \text{ in } \Omega_{\pi,0},\ \ \mathbf{v} = 0 \ \text{ on } \Gamma_{\pi,0},\ \ \mathbf{v} = 0 \ \text{ on } \Gamma_i,\\ &v_2 = 0 \ \text{ and } q = (\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{n})v_1,\ \text{ on } \Gamma_o,\ \ \mathbf{v} = 0 \ \text{ on } \Gamma_b.\end{aligned}$$

Finally, remark that $q - L_o(\mathbf{v}) \in H^1_{\Gamma_o}(\Omega_{\pi,0})$ and so $\Pi\nabla q = \Pi\nabla L_o(\mathbf{v})$. $\square$

Remark 4.4.2. Thanks to the choice of λ_0 in (4.4.2) the adjoint equation of (4.4.1) satisfies the same regularity properties and estimates as the primal system.

We introduce the lifting operator $L \in \mathcal{L}(H^1_{\text{lift}}(\Omega_{\pi,0}) \times \mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0}), \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0}))$ defined by

$$L(w,\mathbf{g}) = (L_1(w,\mathbf{g}), L_2(w,\mathbf{g})) = (\mathbf{u},p), \tag{4.4.16}$$

where $(\mathbf{u},p)$ is the solution to (4.4.1) with $\mathbf{f} = 0$ and $\mathbf{u}_c = 0$. We also introduce $L_{\Gamma_i} \in \mathcal{L}(\mathbf{H}^{3/2}(\Gamma_i), \mathbf{H}^2(\Omega_{\pi,0}))$ defined by

$$L_{\Gamma_i}(\mathbf{u}_c) = \mathbf{u}, \tag{4.4.17}$$

where $(\mathbf{u},p)$ is the solution to (4.4.1) with $\mathbf{f} = 0$, $w = 0$ and $\mathbf{g} = 0$.

Theorem 4.4.4. A pair $(\mathbf{u}, q) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ is solution of system (4.4.1) if and only if

$$\begin{aligned}
&(\lambda_0 I - A(t))\Pi\mathbf{u} + (A(t) - \lambda_0 I)\Pi L_1(w, \mathbf{g}) + (A(t) - \lambda_0 I)\Pi L_{\Gamma_i}(\mathbf{u}_c) = \Pi\mathbf{f}, \\
&(I - \Pi)\mathbf{u} = \nabla N_s(\mathbf{g}) + \nabla N_{\mathrm{div}}(w) + \nabla N_i(\mathbf{u}_c), \\
&q = -\lambda_0 N_{\mathrm{div}}(w) - \lambda_0 N_s(\mathbf{g}) - \lambda_0 N_i(\mathbf{u}_c) + N_v(\mathbf{u}).
\end{aligned} \tag{4.4.18}$$

Proof. For the fluid part, remark that $\hat{\mathbf{u}} = \mathbf{u} - L_1(w, \mathbf{g}) - L_{\Gamma_i}(\mathbf{u}_c)$ is in $\mathcal{D}(A(t))$ and, applying Π to the first equation of (4.4.1),

$$\lambda_0 \hat{\mathbf{u}} - A(t)\Pi\hat{\mathbf{u}} = \Pi\mathbf{f},$$

where we have used that $\Pi\hat{\mathbf{u}} = \hat{\mathbf{u}}$. Then we obtain

$$(\lambda_0 I - A(t))\Pi\mathbf{u} + (A(t) - \lambda_0 I)\Pi L_1(w, \mathbf{g}) + (A(t) - \lambda_0 I)\Pi L_{\Gamma_i}(\mathbf{u}_c) = \Pi\mathbf{f}.$$

By definition, $(I - \Pi)\mathbf{u} = \nabla p_v$ where p_v is solution to the equation

$$\begin{aligned}
&\Delta p_v = \mathrm{div}\ \mathbf{u} = w \ \text{ in } \Omega_{\pi,0}, \\
&\frac{\partial p_v}{\partial \mathbf{n}} = \mathbf{g} \cdot \mathbf{n} \ \text{ on } \Gamma_{\pi,0}, \\
&\frac{\partial p_v}{\partial \mathbf{n}} = \mathbf{u}_c \cdot \mathbf{n} \ \text{ on } \Gamma_i, \\
&\frac{\partial p_v}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_b, \\
&p_v = 0 \ \text{ on } \Gamma_o,
\end{aligned} \tag{4.4.19}$$

that can be split in the three equations (4.4.15), (4.4.13) and (4.4.14). Therefore we obtain $p_v = N_{\mathrm{div}}(w) + N_s(\mathbf{g}) + N_i(\mathbf{u}_c)$. The expression of the pressure was proved in Lemma 4.4.2. $\square$

4.5 The Linearized system

4.5.1 Stationary system

In this section we study a stationary system that is used later to reformulate the (4.2.9) as a matrix evolution equation.

Remark 4.5.1. The 'stationary' system (for each t) that we study in this section is different from the stationary system directly associated with (4.2.9), that is where $(\mathbf{v}_t, \hat{\eta}_{1,t}, \hat{\eta}_{2,t})$ is replaced by $\lambda(\mathbf{v}, \hat{\eta}_1, \hat{\eta}_2)$. The reason for this difference comes from the time dependency of the term A_3 in the divergence of the fluid. Precisely, the difference between the two is the additional term $N_{\mathrm{div}}(A_{3,t}\hat{\eta}_1)$ in the beam equation. This term does not appear in the usual stationary system associated with (4.2.9) but appears in the evolution problem, because of the replacement of the pressure by its expression in terms of the velocity.

For $\lambda \in \mathbb{C}$ and $t \geq 0$ (implicit in the coefficients of the following stationnary system), consider

$$
\begin{aligned}
&\lambda \mathbf{u} + D_\pi \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p - A_{1,1}\hat{\eta}_1 - A_2 \hat{\eta}_2 = F_1 \ \text{ in } \Omega_{\pi,0},\\
&\text{div } \mathbf{u} = A_3 \hat{\eta}_1 \ \text{ in } \Omega_{\pi,0},\\
&\mathbf{u} = \hat{\eta}_2 \mathbf{e}_2 \ \text{ on } \Gamma_{\pi,0},\\
&\mathbf{u} = 0 \ \text{ on } \Gamma_i,\\
&u_2 = 0 \ \text{ and } \ p = 0 \text{ on } \Gamma_o,\\
&\mathbf{u} = 0 \ \text{ on } \Gamma_b,\\
&\lambda \hat{\eta}_1 - \hat{\eta}_2 = F_2 \text{ on } \Gamma_s,\\
&\lambda \hat{\eta}_2 - \beta \hat{\eta}_{1,xx} - \gamma \hat{\eta}_{2,xx} + \alpha \hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1 - A_5 \hat{\eta}_1\\
&\qquad = -J_{\eta^0_{\pi,1}} \mathbf{e}_2 \cdot \sigma(\mathbf{u}, p)_{|\Gamma_{\eta^0_{\pi,1}}} \mathbf{n}_{\eta^0_{\pi,1}} + F_3 \ \text{ in } \Gamma_s.\\
&\hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0, L\},
\end{aligned}
\tag{4.5.1}
$$

with $(F_1, F_2, F_3) \in \mathbf{L}^2(\Omega_{\pi,0}) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s)$ and A_5 a linear operator, described below, on $\hat{\eta}_1$ that corresponds to the term $N_{\text{div}}(A_{3,t}\hat{\eta}_1)$ appearing in the pressure of the evolution problem. The boundary $\Gamma_{\pi,0}$ may not be flat, hence we use the techniques introduced in Chapter 2. Consider the transport operator $\mathcal{U} \in \mathcal{L}(L^2(\Gamma_{\pi,0}), L^2(\Gamma_s))$ defined by

$$(\mathcal{U}g)(x,1) = g(x, 1 + \eta^0_{\pi,1}(x)) \quad \text{for all } g \in L^2(\Gamma_{\pi,0}).$$

The operator $\mathcal{U}$ is an isomorphism from $L^2(\Gamma_{\pi,0})$ to $L^2(\Gamma_s)$ which satisfies $\mathcal{U}^{-1} = \mathcal{U}^*$ (see Chapter 2 for details).

In order to express the pressure on Γ_s we also introduce the following operators:

- $\overline{N}_s(\eta) = \mathcal{U} N_s(\mathcal{U}^* \eta \mathbf{e}_2)$.
- $\overline{N}_{\text{div}}(w) = \mathcal{U} N_{\text{div}}(w)$.
- $\overline{N}_v(\mathbf{u}) = \mathcal{U} N_v(\mathbf{u})$.
- $\overline{N}_p = \mathcal{U} N_p$.

The linear operator A_5 is given by

$$A_5 \hat{\eta}_1 = -\overline{N}_{\text{div}}(A_{3,t}\hat{\eta}_1).$$

Lemma 4.5.1. The differential operators $A_{1,1}$, A_2, A_3, $A_{4,1}$ and A_5 satisfy

$$\begin{array}{ll} A_{1,1} \in \mathcal{C}^{\rho}_{\sharp}([0,T];\mathcal{L}(H^2_0(\Gamma_s),\mathbf{L}^2(\Omega_{\pi,0})))\,, & A_2 \in \mathcal{C}^{\rho}_{\sharp}([0,T];\mathcal{L}(L^2(\Gamma_s),\mathbf{L}^2(\Omega_{\pi,0}))), \\ A_3 \in \mathcal{C}^{1+\rho}_{\sharp}([0,T];\mathcal{L}(H^2_0(\Gamma_s),H^1(\Omega_{\pi,0})))\,, & A_{4,1} \in \mathcal{C}^{\rho}_{\sharp}([0,T];\mathcal{L}(H^2_0(\Gamma_s),H^{1/2}(\Gamma_s))), \\ A_5 \in \mathcal{C}^{\rho}_{\sharp}([0,T];\mathcal{L}(H^2_0(\Gamma_s),H^{5/2}(\Gamma_s))). & \end{array}$$

Proof. These results rely on direct estimates and Soboblev embeddings using the regularity of the periodic solution $(\overline{\mathbf{u}}_\pi, \overline{p}_\pi, \eta_\pi)$ (see Section 4.2.2). $\square$

For $t \in (0,T)$, the dual operators of $A_{1,1}(t)$, $A_2(t)$, $A_3(t)$ and $A_{4,1}(t)$ are denoted by

$$\begin{array}{ll} A^*_{1,1}(t) \in \mathcal{L}(\mathbf{L}^2(\Omega_{\pi,0}),H^{-2}(\Gamma_s))\,, & A^*_2(t) \in \mathcal{L}(\mathbf{L}^2(\Omega_{\pi,0}),L^2(\Gamma_s)), \\ A^*_3(t) \in \mathcal{L}(L^2(\Omega_{\pi,0}),H^{-2}(\Gamma_s))\,, & A^*_{4,1}(t) \in \mathcal{L}(L^2(\Gamma_s),H^{-2}(\Gamma_s)), \\ A^*_5(t) \in \mathcal{L}(L^2(\Gamma_s),H^{-2}(\Gamma_s)), & \end{array}$$

where $H^{-2}(\Gamma_s)$ is the dual space of $H^2_0(\Gamma_s)$.

4.5.2 The family of operators $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{t\geq 0}$

Let $(A_{\alpha,\beta}, \mathcal{D}(A_{\alpha,\beta}))$ be the unbounded operator in $L^2(\Gamma_s)$ defined by $\mathcal{D}(A_{\alpha,\beta}) = H^4(\Gamma_s) \cap H^2_0(\Gamma_s)$ and, for all $\eta \in \mathcal{D}(A_{\alpha,\beta})$, $A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha\eta_{xxxx}$. The operator $A_{\alpha,\beta}$ is self-adjoint and is an isomorphism from $\mathcal{D}(A_{\alpha,\beta})$ to $L^2(\Gamma_s)$. It can be extended by duality as an isomorphism from $L^2(\Gamma_s)$ to $\mathcal{D}(A_{\alpha,\beta})'$ and using interpolation $A_{\alpha,\beta} \in \mathcal{L}(H^{7/2}(\Gamma_s) \cap H^2_0(\Gamma_s), H^{-1/2}(\Gamma_s))$.

The space $H^2_0(\Gamma_s)$ is equipped with the inner product

$$\langle \eta_1, k_1 \rangle_{H^2_0(\Gamma_s)} = \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1 (-A_{\alpha,\beta})^{1/2} k_1.$$

We also introduce the spaces

$$\hat{\mathbf{H}} = \mathbf{L}^2(\Omega_{\pi,0}) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s), \;\; \mathbf{H} = \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s),$$

both equipped with the inner product

$$\langle(\mathbf{u},\eta_1,\eta_2),(\mathbf{v},\zeta_1,\zeta_2)\rangle_{\hat{\mathbf{H}}} = \langle \mathbf{u},\mathbf{v}\rangle_{\mathbf{L}^2(\Omega_{\pi,0})} + \langle \eta_1,\zeta_1\rangle_{H^2_0(\Gamma_s)} + \langle \eta_2,\zeta_2\rangle_{L^2(\Gamma_s)}.$$

The unbounded operator $(A_b, \mathcal{D}(\mathcal{A}_b))$ related to the beam, in $H_b = H^2_0(\Gamma_s) \times L^2(\Gamma_s)$, is defined by

$$\mathcal{D}(\mathcal{A}_b) = (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s) \text{ and } \mathcal{A}_b = \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} & \gamma\Delta_s \end{pmatrix}.$$

Lastly we introduce the space

$$\mathcal{D}(\mathcal{A}(t)) := \{(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0}) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s) \\ \mid \Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(A(t))\}. \tag{4.5.2}$$

The operator N_v is initially defined with values in $L^2(\Omega_{\pi,0})$. The following lemma shows that we can improved its regularity in special cases.

Lemma 4.5.2. For all $t \geq 0$, the following regularity results hold.

- For all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(\mathcal{A}(t))$,

$$N_v[\Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)] \in H^1(\Omega_{\pi,0}).$$

- For all $0 < \varepsilon < 1/2$ and $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in H^{3/2+\varepsilon}(\Omega_{\pi,0}) \times \Big(H^{7/2+\varepsilon}(\Gamma_s) \cap H^2_0(\Gamma_s)\Big) \times H^{3/2+\varepsilon}_0(\Gamma_s)$,

$$N_v[\Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)] \in L^2(\Omega_{\pi,0}).$$

- There exists $\frac{7}{8} < \kappa < 1$ and $\varepsilon_0 > 0$ such that for all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in [\mathcal{D}(\mathcal{A}(t)), \mathbf{H}]_{1-\kappa}$,

$$N_v[\Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)] \in H^{1/2+\varepsilon_0}(\Omega_{\pi,0}).$$

Proof. The first point is similar to [23, Lemma 5.5]. If $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)$ belongs to $\mathcal{D}(\mathcal{A}(t))$ then there exists $G \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$ such that

$$(\lambda_0 I - A(t))\Pi\mathbf{u} + (A(t) - \lambda_0 I)\Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) = G.$$

Setting $\mathbf{u} = \Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)$ and $p = -\lambda_0 N_{\mathrm{div}}(A_3\hat{\eta}_2) - \lambda_0 N_s(\hat{\eta}_2) + N_v(\mathbf{u})$ we remark that $(\mathbf{u}, p)$ is the solution to the system

$$\begin{aligned} &\lambda_0\mathbf{u} + D_\pi(t)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = G \ \text{ in } \Omega_{\pi,0}, \\ &\mathrm{div}\ \mathbf{u} = A_3\hat{\eta}_1 \ \text{ in } \Omega_{\pi,0}, \ \ \mathbf{u} = \hat{\eta}_2\mathbf{e}_2 \ \text{ on } \Gamma_{\pi,0}, \ \ \mathbf{u} = 0 \ \text{ on } \Gamma_i, \\ &u_2 = 0 \ \text{ and } p = 0 \ \text{ on } \Gamma_o, \ \ \mathbf{u} = 0 \ \text{ on } \Gamma_b. \end{aligned}$$

Using the regularity result in Theorem 4.4.1 we deduce that $p \in H^1(\Omega_{\pi,0})$. Finally, noticing that $A_3\hat{\eta}_1 \in H^1_{\Gamma_o}(\Omega_{\pi,0})$ and using Lemma 4.4.3, the terms $\lambda_0 N_{\mathrm{div}}(A_3\hat{\eta}_2) - \lambda_0 N_s(\eta_2)$ and $N_v(\mathbf{u}) = p + \lambda_0 N_{\mathrm{div}}(A_3\hat{\eta}_2) - \lambda_0 N_s(\hat{\eta}_2)$ belong to $H^1(\Omega_{\pi,0})$.

For the second point we use Lemma 4.4.3 to show that $N_s(\hat{\eta}_2) \in H^3(\Omega_{\pi,0})$ and $N_{\mathrm{div}}(A_3\hat{\eta}_1) \in H^3(\Omega_{\pi,0})$. Hence $\Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)$ belongs to $H^{3/2+\varepsilon_0}(\Omega_{\pi,0})$ and we use the the definition of N_v to conclude.

Finally, the last point comes from interpolation techniques. $\square$

Theorem 4.5.1. A pair $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ obeys the fluid equation in (4.5.1) if and only if

$$\begin{aligned}
&\lambda \Pi \mathbf{u} - A(t)\Pi \mathbf{u} + (A(t) - \lambda_0 I)\Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) - \Pi A_{1,1}\hat{\eta}_1 - \Pi A_2\hat{\eta}_2 = \Pi F_1,\\
&(I - \Pi)\mathbf{u} = \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1),\\
&p = -\lambda N_{\mathrm{div}}(A_3\hat{\eta}_1) - \lambda N_s(\hat{\eta}_2) + N_v(\Pi \mathbf{u}) + N_v(\nabla N_{\mathrm{div}}(A_3\hat{\eta}_1) + \nabla N_s(\hat{\eta}_2))\\
&\quad + N_p(A_{1,1}\hat{\eta}_1) + N_p(A_2\hat{\eta}_2) + N_p(F_1),
\end{aligned}$$

where $N_s(\hat{\eta}_2) := N_s(\mathcal{U}\hat{\eta}_2\mathbf{e}_2)$.

We introduce the operator $B_\nabla \in \mathcal{L}(\mathbf{H}^2(\Omega_{\pi,0}), H^{1/2}(\Gamma_s))$,

$$B_\nabla(\mathbf{u}) = -J_{\eta^0_{\pi,1}} \mathbf{e}_2 \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T)_{|\Gamma_{\eta^0_{\pi,1}}} \mathbf{n}_{\eta^0_{\pi,1}}.$$

The beam equation in (4.5.1) becomes

$$\begin{aligned}
&\lambda\hat{\eta}_1 - \hat{\eta}_2 = F_2,\\
&\lambda(I + \overline{N}_s(\cdot))\hat{\eta}_2 + \lambda\overline{N}_{\mathrm{div}}(A_3\hat{\eta}_1) - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1 - A_5\hat{\eta}_1\\
&\quad = B_\nabla(\Pi\mathbf{u}) + B_\nabla[\nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)] + \overline{N}_v[\Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)]\\
&\qquad + \overline{N}_p(A_{1,1}\hat{\eta}_1) + \overline{N}_p(A_2\hat{\eta}_2) + \overline{N}_p(F_1) + F_3,
\end{aligned}$$

which we can rewrite as

$$\lambda \begin{pmatrix} I & 0 \\ \overline{N}_{\mathrm{div}}(A_3\cdot) & (I + \overline{N}_s(\cdot)) \end{pmatrix} \begin{pmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} = \left(\begin{array}{c} \hat{\eta}_2 + F_2 \\ \hline A_{\alpha,\beta}\hat{\eta}_1 + \gamma\Delta\hat{\eta}_2 + (A_{4,1} + A_5)\hat{\eta}_1 + B_\nabla(\Pi\mathbf{u}) \\ + B_\nabla(\nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)) \\ + \overline{N}_v(\Pi\mathbf{u} + \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1)) \\ + \overline{N}_p(A_{1,1}\hat{\eta}_1) + \overline{N}_p(A_2\hat{\eta}_2) + \overline{N}_p(F_1) + F_3 \end{array} \right).$$

Let us introduce the 'added mass' operator $M_s \in \mathcal{L}(\mathbf{H})$ defined by

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \overline{N}_{\mathrm{div}}(A_3\cdot) & (I + \overline{N}_s(\cdot)) \end{pmatrix}. \tag{4.5.3}$$

Lemma 4.5.3. The operator M_s is an automorphism in $\mathbf{H}$ and

$$M_s^{-1} := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -(I + \overline{N}_s(\cdot))^{-1}\overline{N}_{\mathrm{div}}(A_3\cdot) & (I + \overline{N}_s(\cdot))^{-1} \end{pmatrix}.$$

Proof. As in Chapter 2 the operator $(I + \overline{N}_s)$ is an automorphism in $L^2(\Gamma_s)$. The result follows. ∎

Introduce the family of operators $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t))_{t\geq 0}$ where $\mathcal{D}(\mathcal{A}(t))$ is defined by (4.5.2) and $\mathcal{A}(t) = \mathcal{A}_1(t) + \mathcal{A}_2(t)$ with

$$\mathcal{A}_1(t) := M_s^{-1} \begin{pmatrix} A(t) & (\lambda_0 I - A(t))\Pi L_1(A_3\cdot, 0) & (\lambda_0 I - A(t))\Pi L_1(0,\cdot) \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \gamma\Delta \end{pmatrix}, \tag{4.5.4}$$

and

$$\mathcal{A}_2(t) := M_s^{-1} \left(\begin{array}{c|c|c} 0 & \Pi A_{1,1} & \Pi A_2 \\ \hline 0 & 0 & 0 \\ \hline \overline{N}_v(\cdot) + B_\nabla(\cdot) & \begin{array}{c} A_{4,1} + A_5 + B_\nabla(\nabla N_{\mathrm{div}}(A_3\cdot)) \\ \overline{N}_v(\nabla N_{\mathrm{div}}(A_3\cdot)) + \overline{N}_p(A_{1,1}\cdot) \end{array} & \begin{array}{c} B_\nabla(\nabla N_s(\cdot)) \\ +\,\overline{N}_v(\nabla N_s(\cdot)) \\ +\,\overline{N}_p(A_2\cdot) \end{array} \end{array}\right). \tag{4.5.5}$$

For strong solutions, system (4.5.1) can be rewritten in terms of $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)$,

$$\begin{cases} \lambda \begin{pmatrix} \Pi\mathbf{u} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} \Pi\mathbf{u} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} + M_s^{-1} \begin{pmatrix} \Pi F_1 \\ F_2 \\ \overline{N}_p(F_1) + F_3 \end{pmatrix} \\ (I - \Pi)\mathbf{u} = \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1), \\ p = -\lambda N_{\mathrm{div}}(A_3\hat{\eta}_1) - \lambda N_s(\hat{\eta}_2) + N_v(\mathbf{u}) \\ \qquad + N_p(A_{1,1}\hat{\eta}_1) + N_p(A_2\hat{\eta}_2) + N_p(F_1). \end{cases} \tag{4.5.6}$$

4.5.3 Adjoint of $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t))_{t\geq 0}$

In order to compute the adjoint of $\mathcal{A}(t)$ consider the system, for $\lambda \in \mathbb{C}$ and $t \geq 0$ (implicit in the coefficients of the system),

$$\begin{aligned} &\lambda\mathbf{u} + D_\pi\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p - A_{1,1}\hat{\eta}_1 - A_2\hat{\eta}_2 = F_1 \ \text{ in } \Omega_{\pi,0}, \\ &\mathrm{div}\ \mathbf{u} = A_3\hat{\eta}_1 \ \text{ in } \Omega_{\pi,0}, \\ &\mathbf{u} = \hat{\eta}_2\mathbf{e}_2 \ \text{ on } \Gamma_{\pi,0}, \\ &\mathbf{u} = 0 \ \text{ on } \Gamma_i, \\ &u_2 = 0 \ \text{ and } \ p = 0 \text{ on } \Gamma_o, \\ &\mathbf{u} = 0 \ \text{ on } \Gamma_b, \\ &\lambda\hat{\eta}_1 - \hat{\eta}_2 = F_2 \text{ on } \Gamma_s, \\ &\lambda\hat{\eta}_2 - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1 - A_5\hat{\eta}_1 \\ &\qquad = -J_{\eta^0_{\pi,1}}\mathbf{e}_2 \cdot \sigma(\mathbf{u}, p)_{|\Gamma_{\eta^0_{\pi,1}}}\mathbf{n}_{\eta^0_{\pi,1}} + F_3 \ \text{ in } \Gamma_s. \\ &\hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0, L\}, \end{aligned} \tag{4.5.7}$$

with $(F_1, F_2, F_3) \in \hat{\mathbf{H}}$.

Lemma 4.5.4. Let $(F_1, F_2, F_3) \in \hat{\mathbf{H}}$ and

$$(\mathbf{u}, p, \hat{\eta}_1, \hat{\eta}_2) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega) \times \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s),$$

be the solution to (4.5.7). Then the following identity holds

$$\left(\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ k_1 \\ k_2 \end{pmatrix} \right)_{\hat{\mathbf{H}}} = \left(\begin{pmatrix} \mathbf{u} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix}, \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \right)_{\hat{\mathbf{H}}} \tag{4.5.8}$$

for all $(\Phi_1, \Phi_2, \Phi_3) \in \hat{\mathbf{H}}$ and

$$(\mathbf{v}, q, k_1, k_2) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega) \times \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s)$$

solution to
(4.5.9)

$$\begin{aligned}
&\lambda \mathbf{v} - \nu \Delta \mathbf{v} + D_\pi^a(t)\mathbf{v} + \nabla q = \Phi_1 \ \text{ in } \Omega_{\pi,0}, \\
&\text{div } \mathbf{v} = 0 \ \text{ in } \Omega_{\pi,0}, \\
&\mathbf{v} = k_2 \mathbf{e}_2 \ \text{ on } \Gamma_{\pi,0}, \\
&\mathbf{v} = 0 \ \text{ on } \Gamma_i, \\
&v_2 = 0 \ \text{ and } \ q = (\overline{\mathbf{u}}_\pi(t) \cdot \mathbf{n}) v_1 \text{ on } \Gamma_o, \\
&\mathbf{v} = 0 \ \text{ on } \Gamma_b, \\
&\lambda k_1 + k_2 - (-A_{\alpha,\beta})^{-1} A_{1,1}^* \mathbf{v} - (-A_{\alpha,\beta})^{-1}(-A_3^* + A_{4,1}^* + A_5^*) k_2 + (-A_{\alpha,\beta})^{-1} A_3^* q \\
&\qquad = \Phi_2 \text{ on } \Gamma_s, \\
&(\lambda + \eta_\pi + C_\pi(t)) k_2 + \beta k_{1,xx} - \gamma k_{2,xx} - \alpha k_{1,xxxx} - A_2^* \mathbf{v} + J_{\eta^0_{\pi,1}} \mathbf{e}_2 \cdot \sigma(\mathbf{v}, q)_{|\Gamma_{\eta^0_{\pi,1}}} \mathbf{n}_{\eta^0_{\pi,1}} \\
&\qquad = \Phi_3 \ \text{ in } \Gamma_s. \\
&k_1 = 0 \ \text{ and } \ k_{1,x} = 0 \ \text{ on } \{0, L\}.
\end{aligned}$$

Proof. The proof is based on integration by parts. We come back to the stress tensor notation through the identity

$$-\nu \Delta \mathbf{u} + \nabla p = -\text{div } \sigma(\mathbf{u}, p) + \nu \nabla \text{div } \mathbf{u} = -\text{div } \sigma(\mathbf{u}, p) + \nu \nabla A_3 \hat{\eta}_1.$$

As a preliminary, remark that for a function $f \in L^1(\Gamma_{\pi,0})$

$$\int_{\Gamma_{\pi,0}} f \mathbf{e}_2 \cdot \mathbf{n} = \int_{\Gamma_s} \mathcal{U} f.$$

To simplify the presentation we split F_1 into two parts

$$F_1 = \underbrace{\lambda \mathbf{u} - \text{div } \sigma(\mathbf{u}, p)}_{F_{1,1}} + \underbrace{\nu \nabla A_3 \hat{\eta}_1 + D_\pi \mathbf{u} - A_{1,1} \hat{\eta}_1 - A_2 \hat{\eta}_2}_{F_{1,2}}.$$

Step 1: Using the Green formula, the tensor part in $F_{1,1} \cdot \mathbf{v}$ becomes

$$
\begin{aligned}
-\int_{\Omega_{\pi,0}} \operatorname{div}\, \sigma(\mathbf{u},p)\cdot \mathbf{v} &= \int_{\Omega_{\pi,0}} \sigma(\mathbf{u},p) : \nabla \mathbf{v} - \int_{\partial\Omega_{\pi,0}} (\sigma(\mathbf{u},p)\,\mathbf{n})\cdot \mathbf{v} \\
&= \int_{\Omega_{\pi,0}} \nabla \mathbf{u} : \sigma(\mathbf{v},q) - \frac{1}{2\nu}\int_{\Omega_{\pi,0}} pI : \sigma(\mathbf{v},q) \\
&\quad + \frac{1}{2\nu}\int_{\Omega_{\pi,0}} qI : \sigma(\mathbf{u},p) - \int_{\partial\Omega_{\pi,0}} (\sigma(\mathbf{u},p)\,\mathbf{n})\cdot \mathbf{v},
\end{aligned}
$$

where we have used the symmetry of $\sigma(\mathbf{u},p)$ (respectively of $\sigma(\mathbf{v},q)$) to write

$$
\sigma(\mathbf{u},p) : \nabla \mathbf{v} = \frac{1}{2}\sigma(\mathbf{u},p) : (\nabla \mathbf{v} + \nabla \mathbf{v}^T).
$$

Moreover

$$
\begin{aligned}
\frac{1}{2\nu}\int_{\Omega_{\pi,0}} pI : \sigma(\mathbf{v},q) - \frac{1}{2\nu}\int_{\Omega_{\pi,0}} qI : \sigma(\mathbf{u},p) &= \int_{\Omega_{\pi,0}} p \operatorname{div}\, \mathbf{v} - \int_{\Omega_{\pi,0}} q \operatorname{div}\, \mathbf{u} \\
= -\int_{\Omega_{\pi,0}} q\, A_3\hat{\eta}_1 = -\langle \hat{\eta}_1, A_3^* q\rangle_{H_0^2(\Gamma_s),H^{-2}(\Gamma_s)} &= -\langle \hat{\eta}_1, (-A_{\alpha,\beta})^{-1}A_3^* q\rangle_{H_0^2(\Gamma_s)}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
-\int_{\Omega_{\pi,0}} \operatorname{div}\, \sigma(\mathbf{u},p)\cdot \mathbf{v} = & -\int_{\Omega_{\pi,0}} \mathbf{u}\cdot \operatorname{div}\, \sigma(\mathbf{v},q) - \int_{\partial\Omega_{\pi,0}} (\sigma(\mathbf{u},p)\,\mathbf{n})\cdot \mathbf{v} \\
& + \int_{\partial\Omega_{\pi,0}} (\sigma(\mathbf{v},q)\,\mathbf{n})\cdot \mathbf{u} + \langle \hat{\eta}_1, (-A_{\alpha,\beta})^{-1}A_3^* q\rangle_{H_0^2(\Gamma_s)}.
\end{aligned}
$$

Moreover the terms $(\sigma(\mathbf{u},p)\,\mathbf{n})\cdot\mathbf{v}$ and $(\sigma(\mathbf{v},q)\,\mathbf{n})\cdot\mathbf{u}$ vanish on $\Gamma_i\cup\Gamma_b$ as $\mathbf{u}=\mathbf{v}=0$ on $\Gamma_i\cup\Gamma_b$. On Γ_o, since $v_2=0$ and $p=0$, we have

$$
(\sigma(\mathbf{u},p)\,\mathbf{n})\cdot\mathbf{v} = 2\nu\partial_1 u_1 v_1.
$$

Remark then that, on Γ_o, $\partial_2 u_2 = 0$ and $\partial_1 u_1 = \operatorname{div}\, \mathbf{u} = A_3\hat{\eta}_1 = 0$. Hence $(\sigma(\mathbf{u},p)\,\mathbf{n})\cdot\mathbf{v}$ also vanishes on Γ_o. The same reasoning can be done on $(\sigma(\mathbf{v},q)\,\mathbf{n})\cdot\mathbf{u}$ except that $q\neq 0$ on Γ_o. Hence,

$$
(\sigma(\mathbf{v},q)\,\mathbf{n})\cdot\mathbf{u} = -qu_1 \text{ on } \Gamma_o. \tag{4.5.10}
$$

Using the boundary condition on q this quantity will simplifies with another boundary term on Γ_o coming from the Oseen terms.

Step 2: The term $F_{1,2}$ involves the perturbations, that is,

$$
\int_{\Omega_{\pi,0}} (\nu\nabla A_3\hat{\eta}_1 + D_\pi \mathbf{u} - A_{1,1}\hat{\eta}_1 - A_2\hat{\eta}_2)\cdot\mathbf{v},
$$

with

$$
\int_{\Omega_{\pi,0}} D_\pi \mathbf{u}\cdot\mathbf{v} = \int_{\Omega_{\pi,0}} ((\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{u}}_\pi(t) + C_\pi(t)\mathbf{u}_z)\cdot\mathbf{v}.
$$

Using duality,

$$-\int_{\Omega_{\pi,0}} A_{1,1}\hat{\eta}_1 \cdot \mathbf{v} = -\langle \hat{\eta}_1, A_{1,1}^* \mathbf{v}\rangle_{H_0^2(\Gamma_s),H^{-2}(\Gamma_s)} = -\langle \hat{\eta}_1, (-A_{\alpha,\beta})^{-1} A_{1,1}^* \mathbf{v}\rangle_{H_0^2(\Gamma_s)},$$

and

$$-\int_{\Omega_{\pi,0}} A_2\hat{\eta}_2 \cdot \mathbf{v} = -\int_{\Gamma_s} \hat{\eta}_2 A_2^* \mathbf{v}.$$

For the Oseen terms we have

$$\int_{\Omega_{\pi,0}} (\mathbf{u}\cdot\nabla)\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{v} = \int_{\Omega_{\pi,0}} \mathbf{u}\cdot(\nabla\overline{\mathbf{u}}_\pi(t))^T\mathbf{v},$$

and

$$\begin{aligned}\int_{\Omega_{\pi,0}} (\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{u}\cdot\mathbf{v} = & -\int_{\Omega_{\pi,0}} \mathbf{u}\cdot\mathbf{v}\,\mathrm{div}\,(\overline{\mathbf{u}}_\pi(t)) - \int_{\Omega_{\pi,0}} \mathbf{u}\cdot(\overline{\mathbf{u}}_\pi(t)\cdot\nabla)\mathbf{v} \\ & + \int_{\partial\Omega_{\pi,0}} (\mathbf{u}\cdot\mathbf{v})\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{n},\end{aligned}$$

with

$$\int_{\partial\Omega_{\pi,0}} (\mathbf{u}\cdot\mathbf{v})\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{n} = \int_{\Gamma_o} (\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{n})v_1u_1 + \int_{\Gamma_{\pi,0}} k_2\hat{\eta}_2\eta_\pi\mathbf{e}_2\cdot\mathbf{n}.$$

Using the boundary condition on q, the first term simplifies with (4.5.10). For the second term

$$\int_{\Gamma_{\pi,0}} k_2\hat{\eta}_2\eta_\pi\mathbf{e}_2\cdot\mathbf{n} = \int_{\Gamma_s} k_2\hat{\eta}_2\eta_\pi.$$

Finally,

$$\int_{\Omega_{\pi,0}} C_\pi(t)\mathbf{u}_z\cdot\mathbf{v} = -\int_{\Omega_{\pi,0}} \mathbf{u}\cdot(C_{\pi,z}(t)\mathbf{v} + C_\pi(t)\mathbf{v}_z) + \int_{\partial\Omega_{\pi,0}} (\mathbf{u}\cdot\mathbf{v})C_\pi(t)\mathbf{n}\cdot\mathbf{e}_2,$$

where, with a similar reasoning as above and using that $C_\pi(t) = 0$ on Γ_o,

$$\int_{\partial\Omega_{\pi,0}} (\mathbf{u}\cdot\mathbf{v})C_\pi(t)\mathbf{n}\cdot\mathbf{e}_2 = \int_{\Gamma_s} \hat{\eta}_2 k_2 C_\pi(t).$$

The only term that remains in $F_{1,2}$ is

$$\nu\int_{\Omega_{\pi,0}} \nabla A_3\hat{\eta}_1\cdot\mathbf{v} = -\int_{\Gamma_{\pi,0}} A_3\hat{\eta}_1 k_2\mathbf{e}_2\cdot\mathbf{n} = \int_{\Gamma_s} A_3\hat{\eta}_1 k_2 = \langle\hat{\eta}_1, (-A_{\alpha,\beta})^{-1}A_3^* k_2\rangle_{H_0^2(\Gamma_s)}.$$

If we summarize the previous results we obtain

$$\begin{aligned}\int_{\Omega_{\pi,0}} F_1\cdot\mathbf{v} = & \int_{\Omega_{\pi,0}} \Phi_1\cdot\mathbf{u} - \int_{\Gamma_{\pi,0}} (\sigma(\mathbf{u},p)\,\mathbf{n})\cdot\mathbf{v} + \int_{\Gamma_{\pi,0}} (\sigma(\mathbf{v},q)\,\mathbf{n})\cdot\mathbf{u} \\ & + \langle\hat{\eta}_1, (-A_{\alpha,\beta})^{-1}A_3^*(q+k_2)\rangle_{H_0^2(\Gamma_s)} - \langle\hat{\eta}_1, (-A_{\alpha,\beta})^{-1}A_{1,1}^*\mathbf{v}\rangle_{H_0^2(\Gamma_s)} \\ & - \int_{\Gamma_s} \hat{\eta}_2 A_2^*\mathbf{v} + \int_{\Gamma_s} \hat{\eta}_2(\eta_\pi + C_\pi(t))k_2.\end{aligned}$$

Step 3: For the structure,

$$\langle F_2, k_1\rangle_{H^2_0(\Gamma_s)} = \langle \hat{\eta}_1, \lambda k_1\rangle_{H^2_0(\Gamma_s)} - \int_{\Gamma_s} \hat{\eta}_2 A_{\alpha,\beta} k_1,$$

and

$$\begin{aligned}
\int_{\Gamma_s} F_3 k_2 &= \lambda \int_{\Gamma_s} \hat{\eta}_2 k_2 - \beta \int_{\Gamma_s} \hat{\eta}_{1,xx} k_2 - \gamma \int_{\Gamma_s} \hat{\eta}_{2,xx} k_2 + \alpha \int_{\Gamma_s} \hat{\eta}_{1,xxxx} k_2 \\
&\quad - \int_{\Gamma_s} (A_{4,1} + A_5)\hat{\eta}_1 k_2 + \int_{\Gamma_s} J_{\eta^0_{\pi,1}} \mathbf{e}_2 \cdot \sigma(\mathbf{u}, p)_{|\Gamma_{\eta^0_{\pi,1}}} \mathbf{n}_{\eta^0_{\pi,1}} k_2 \\
&= \int_{\Gamma_s} \hat{\eta}_2 (\lambda k_2 - \gamma k_{2,xx}) + \langle \hat{\eta}_1, k_2\rangle_{H^2_0(\Gamma_s)} - \langle \hat{\eta}_1, (-A_{\alpha,\beta})^{-1} A^*_{4,1} k_2\rangle_{H^2_0(\Gamma_s)} \\
&\quad + \int_{\Gamma_s} J_{\eta^0_{\pi,1}} \mathbf{e}_2 \cdot \sigma(\mathbf{u}, p)_{|\Gamma_{\eta^0_{\pi,1}}} \mathbf{n}_{\eta^0_{\pi,1}} k_2.
\end{aligned}$$

Step 4: Finally, adding all the terms in (4.5.8) and putting $\mathbf{u}$, $\hat{\eta}_1$ and $\hat{\eta}_2$ in factor we obtain

$$\int_{\Omega_{\pi,0}} F_1 \cdot \mathbf{v} + \langle F_2, k_1\rangle_{H^2_0(\Gamma_s)} + \int_{\Gamma_s} F_3 k_2 = \int_{\Omega_{\pi,0}} \Phi_1 \cdot \mathbf{u} + \langle \Phi_2, \hat{\eta}_1\rangle_{H^2_0(\Gamma_s)} + \int_{\Gamma_s} \Phi_3 \hat{\eta}_2,$$

which concludes the proof. $\square$

In order to express the pressure in (4.5.9) we introduce, for $0 < \varepsilon_0 < 1/2$, the operator

$$N_{v*} \in \mathcal{L}(\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\pi,0}), L^2(\Omega_{\pi,0}))$$

defined by $N_{v*}(\mathbf{u}) = q_1$ where q_1 is given by the Riesz representation theorem and the identity, for all $\xi \in L^2(\Omega_{\pi,0})$ and $\chi_\xi \in H^2(\Omega_{\pi,0})$ solution to (4.4.10),

$$\begin{aligned}
\int_{\Omega_{\pi,0}} q_1 \xi &= \nu \int_{\Omega_{\pi,0}} \nabla \mathbf{u} : \nabla^2 \chi_\xi + \nu \int_{\partial\Omega_{\pi,0}} (\nabla \mathbf{u}\, \nabla \chi_\xi) \cdot \mathbf{n} + \int_{\Omega_{\pi,0}} D^a_\pi(t) \mathbf{u} \cdot \nabla \chi_\xi \\
&\quad + \int_{\Gamma_o} (\overline{\mathbf{u}}_\pi(t) \cdot \mathbf{n}) v_1 \nabla \chi_\xi \cdot \mathbf{n}.
\end{aligned}$$

We also introduce the operator $D \in \mathcal{L}(\mathcal{H}^{3/2}_{00}(\Omega_{\pi,0}), \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0}))$ defined by

$$D(\mathbf{g}) = (D_1(\mathbf{g}), D_2(\mathbf{g})) = (\mathbf{w}, \rho),$$

where $(\mathbf{w}, \rho)$ is solution to

$$\begin{aligned}
&\lambda_0 \mathbf{w} - \nu \Delta \mathbf{w} + D^a_\pi(t) \mathbf{w} + \nabla \rho = 0 \ \text{ in } \Omega_{\pi,0}, \\
&\operatorname{div} \mathbf{w} = 0 \ \text{ in } \Omega_{\pi,0}, \ \ \mathbf{w} = 0 \ \text{ on } \Gamma_{\pi,0}, \ \ \mathbf{w} = 0 \ \text{ on } \Gamma_i, \\
&w_2 = 0 \ \text{ and } \rho = (\overline{\mathbf{u}}_\pi(t) \cdot \mathbf{n}) w_1, \ \text{ on } \Gamma_o, \ \ \mathbf{w} = 0 \ \text{ on } \Gamma_b.
\end{aligned}$$

Lemma 4.5.5. The adjoint of the operator M_s in $\mathbf{H}$ is

$$M_s^* = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -(-A_{\alpha,\beta})^{-1}N_s \\ 0 & 0 & (I+\overline{N}_s) \end{pmatrix}$$

and its inverse

$$(M_s^*)^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & (-A_{\alpha,\beta})^{-1}A_3^*N_s(I+\overline{N}_s)^{-1} \\ 0 & 0 & (I+\overline{N}_s)^{-1} \end{pmatrix}$$

Proof. The proof is similar to [23, Lemma 6.1]. □

Lemma 4.5.6. Let $(\Phi_1, \Phi_2, \Phi_3) \in \hat{\mathbf{H}}$, then

$$(\mathbf{v}, q, k_1, k_2) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega) \times \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s),$$

is solution of (4.5.9) if and only if

$$\begin{cases} \lambda M_s^* \begin{pmatrix} \Pi\mathbf{v} \\ k_1 \\ k_2 \end{pmatrix} = M_s^*\mathcal{A}_*(t) \begin{pmatrix} \Pi\mathbf{v} \\ k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} \Pi\Phi_1 \\ \Phi_2 - (-A_{\alpha,\beta})^{-1}A_3^*N_p(\Phi_1) \\ \Phi_3 + \overline{N}_p(\Phi_1) \end{pmatrix} \\ (I-\Pi)\mathbf{v} = \nabla N_s(k_2), \\ q = -\lambda N_s(k_2) + N_{v*}(\mathbf{v}) + N_p(\Phi_1), \end{cases} \tag{4.5.11}$$

where $(\mathcal{A}_*(t), \mathcal{D}(\mathcal{A}_*(t)))_{t\geq 0}$ is the family operators defined by

$$\mathcal{D}(\mathcal{A}_*(t)) := \{(\Pi\mathbf{v}, k_1, k_2) \in \mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0}) \times \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s) \\ \mid \Pi\mathbf{v} - \Pi D_1(k_2) \in \mathcal{D}(A^*(t))\},$$

and $\mathcal{A}_*(t) = \mathcal{A}_*^1(t) + \mathcal{A}_*^2(t)$ with

$$\mathcal{A}_*^1(t) := (M_s^*)^{-1} \begin{pmatrix} A^* & 0 & (\lambda_0 I - A^*)\Pi D_1(\cdot) \\ 0 & 0 & -I \\ 0 & -A_{\alpha,\beta} & \gamma\Delta \end{pmatrix} \tag{4.5.12}$$

and
(4.5.13)

$$\mathcal{A}_*^2(t) := (M_s^*)^{-1} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \begin{array}{c}(-A_{\alpha,\beta})^{-1}A_{1,1}^* \\ -(-A_{\alpha,\beta})^{-1}A_3^*N_v(\cdot)\end{array} & 0 & \begin{array}{l}(-A_{\alpha,\beta})^{-1}A_{1,1}^*\nabla N_s(\cdot) + (-A_{\alpha,\beta})^{-1}A_{4,1}^* \\ +(-A_{\alpha,\beta})^{-1}A_5^* - (-A_{\alpha,\beta})^{-1}A_3^* \\ -(-A_{\alpha,\beta})^{-1}A_3^*N_{v*}(\nabla N_s(\cdot))\end{array} \\ \hline A_2^* + B_\nabla + \overline{N}_{v*} & 0 & \begin{array}{l}(\eta_\pi + C_\pi) + A_2^*\nabla N_s(\cdot) \\ \quad + B_\nabla(\nabla N_s(\cdot)) + \overline{N}_{v*}(\nabla N_s(\cdot))\end{array} \end{array}\right)$$

Proof. Using the same techniques as Theorem 4.5.1 we can prove that $(\mathbf{v}, q) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ is solution to the fluid part of (4.5.9) if and only if

$$\begin{aligned}
&\lambda \mathbf{v} - A^*(t)\Pi\mathbf{v} + (A^*(t) - \lambda_0 I)\Pi D_1(k_2) = \Pi\Phi_1,\\
&(I - \Pi)\mathbf{v} = \nabla N_s(k_2),\\
&q = -\lambda N_s(k_2) + N_{v*}(\mathbf{v}) + N_p(\Phi_1).
\end{aligned}$$

Finally we replace the pressure in the adjoint beam equation to conclude. $\square$

Theorem 4.5.2. The adjoint of the family of operators $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{t\geq 0}$ in $\mathbf{H}$ is the family $(\mathcal{A}^*(t), \mathcal{D}(\mathcal{A}^*(t)))_{t\geq 0}$ defined by

$$\begin{aligned}
\mathcal{D}(\mathcal{A}^*(t)) = \{(\mathbf{v}, k_1, k_2) \in \mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0}) \times \left(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\right) \times H^2_0(\Gamma_s)\\
\mid \Pi\mathbf{v} - \Pi D_1((I + \overline{N}_s)^{-1}k_2) \in \mathcal{D}(A^*(t))\},
\end{aligned} \tag{4.5.14}$$

and

$$\mathcal{A}^*(t) = M_s^* \mathcal{A}_*(t)(M_s^*)^{-1}.$$

Proof. To compute the adjoint of $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t))_{t\geq 0}$ we use the identity (4.5.8) for data in $\mathbf{H}$. Precisely let (F_1, F_2, F_3) be in $\mathbf{H}$ and $(\mathbf{u}, p, \hat{\eta}_1, \hat{\eta}_2) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0}) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s)$ be the solution to (4.5.7). For $(\mathbf{v}, k_1, k_2) \in \mathcal{D}(\mathcal{A}^*(t))$, (4.5.8) becomes

$$\left(\begin{pmatrix} F_1\\ F_2\\ F_3\end{pmatrix}, \begin{pmatrix} \Pi\mathbf{v}\\ k_1\\ k_2\end{pmatrix}\right)_{\mathbf{H}} = \left(\begin{pmatrix} \Pi\mathbf{u}\\ \hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix}, \begin{pmatrix} \Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix}\right)_{\mathbf{H}}.$$

We then use (4.5.6), with $\Pi F_1 = F_1$ and $\overline{N}_p(F_1) = 0$ (since $F_1 \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$),

$$\begin{pmatrix} F_1\\ F_2\\ F_3\end{pmatrix} = \lambda M_s \begin{pmatrix} \Pi\mathbf{u}\\ \hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix} - M_s\mathcal{A}(t)\begin{pmatrix} \Pi\mathbf{u}\\ \hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix},$$

and (4.5.11), with $\Pi\Phi_1 = \Phi_1$ and $N_p(\Phi_1) = \overline{N}_p(\Phi_1) = 0$ (since $\Phi_1 \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$),

$$\begin{pmatrix} \Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} = \lambda M_s^* \begin{pmatrix} \Pi\mathbf{v}\\ k_1\\ k_2\end{pmatrix} - M_s^*\mathcal{A}_*(t)\begin{pmatrix} \Pi\mathbf{v}\\ k_1\\ k_2\end{pmatrix}.$$

Replacing these identities in (4.5.8) we obtain

$$\begin{aligned}
&\left(\lambda M_s \begin{pmatrix} \Pi\mathbf{u}\\ \hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix} - M_s\mathcal{A}(t)\begin{pmatrix} \Pi\mathbf{u}\\ \hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix}, \begin{pmatrix} \Pi\mathbf{v}\\ k_1\\ k_2\end{pmatrix}\right)_{\mathbf{H}}\\
&= \left(\begin{pmatrix} \Pi\mathbf{u}\\ \hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix}, \lambda M_s^* \begin{pmatrix} \Pi\mathbf{v}\\ k_1\\ k_2\end{pmatrix} - M_s^*\mathcal{A}_*(t)\begin{pmatrix} \Pi\mathbf{v}\\ k_1\\ k_2\end{pmatrix}\right)_{\mathbf{H}}.
\end{aligned}$$

Hence $(\lambda M_s - M_s\mathcal{A}(t))^* = \lambda M_s^* - M_s^*\mathcal{A}_*(t)$ so $(M_s\mathcal{A}(t))^* = M_s^*\mathcal{A}_*(t)$. $\square$

4.6 Existence of the evolution operator for $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{t\geq 0}$

In this section we prove the existence of a unique parabolic evolution operator for the family $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{t\geq 0}$. Recall that, even though $\mathcal{D}(A(t)) = V \cap \mathbf{H}^2(\Omega_{\pi,0})$ does not depend on time,

$$\begin{aligned}\mathcal{D}(\mathcal{A}(t)) := \{(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0}) \times (H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \times H^2_0(\Gamma_s) \\ \mid \Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(A(t))\},\end{aligned}$$

genuinely depends on time through A_3 and L_1 that involve the periodic solution at time t. Since Π has co-domain $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$, this compatibility condition only imposes zero values on the tangential trace of $\Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2)$ on $\Gamma_d \cup \Gamma_o$. Now, assume that, for $\theta \in (0,1)$, the compatibility condition in $[\mathcal{D}(\mathcal{A}(t)), \mathbf{H}]_{1-\theta}$ becomes "$\Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) \in [\mathcal{D}(A(t)), \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta}$". If $\theta \in (0, \frac{1}{4})$, then the expected $H^{2\theta}$-regularity of elements in $[\mathcal{D}(A(t)), \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta}$ makes it impossible to define their tangential trace (only a normal trace, already accounted for in $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$), and thus to impose zero values on that trace. We therefore expect, for these θ, that the compatibility condition actually disappears, and that $[\mathcal{D}(A(t)), \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta}$ is independent of t.

This argument is used to construct the parabolic evolution operator for $\mathcal{A}(t)$. However, a balance is required between the Hölder regularity in time and the exponent θ as specified in [4, 2.2, Section IV]. Precisely, the exponent ρ of the periodic solution, which provides the Hölder regularity of $\mathcal{A}(t)$, should be in the interval $(1-\theta, 1)$. From here on we fix a pair

$$(\theta, \rho) \in (0,1)^2 \text{ such that } \theta \in \left(0, \frac{1}{4}\right) \text{ and } \rho \in (1-\theta, 1). \tag{4.6.1}$$

We introduce the space

$$\mathbf{D} = \mathbf{V}^2_{n,\Gamma_d}(\Omega_\pi) \times \Big(H^4(\Gamma_s) \cap H^2_0(\Gamma_s)\Big) \times H^2_0(\Gamma_s), \tag{4.6.2a}$$

equipped with the norm

$$\|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|_{\mathbf{D}} := \left(\|\Pi\mathbf{u}\|^2_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0})} + \|\hat{\eta}_1\|^2_{H^4(\Gamma_s)} + \|\hat{\eta}_2\|^2_{H^2_0(\Gamma_s)}\right)^{1/2}. \tag{4.6.2b}$$

Theorem 4.6.1. There exists $\lambda_2 \in \mathbb{R}$ and $M_2 > 0$ such that $\mathcal{A} : [0,T] \to \mathfrak{A}_{\lambda_2, M_2}(\mathbf{H})$. Moreover the graph norm on $\mathcal{D}(\mathcal{A}(t))$ is uniformly-in-time equivalent to the norm $\|\cdot\|_{\mathbf{D}}$ of $\mathbf{D}$.

Proof. Recalling the definition (4.5.4) of $\mathcal{A}_1(t)$, let us split this operator into $\mathcal{A}_1(t) = \mathcal{A}_{1,1}(t) + \mathcal{A}_{1,2}(t)$ with

$$\mathcal{A}_{1,1}(t) := M_s\mathcal{A}_1(t) = \begin{pmatrix} A(t) & (\lambda_0 I - A(t))\Pi L_1(A_3, 0) & (\lambda_0 I - A(t))\Pi L_1(0, \cdot) \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \gamma\Delta \end{pmatrix},$$

and

$$\mathcal{A}_{1,2}(t) := (M_s^{-1} - I)\mathcal{A}_{1,1}(t).$$

For all $t \geq 0$ we denote by $(\mathcal{D}(\mathcal{A}_{1,1}(t)), \|\cdot\|_{\mathbf{H}} + \|\mathcal{A}_{1,1}(t)\cdot\|_{\mathbf{H}})$ the domain of the operator $\mathcal{A}_{1,1}(t)$, with $\mathcal{D}(\mathcal{A}_{1,1}(t)) := \mathcal{D}(\mathcal{A}(t))$.

Arguing as in [56, Proposition 3.3] and using uniform-in-time estimates on the periodic solution we can prove the existence of $C_1 > 0$ and $C_2 > 0$ independent of t such that, for all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(\mathcal{A}_{1,1}(t))$,

$$\begin{aligned} C_1(\|\Pi\mathbf{u}\|^2_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0})} + \|\hat{\eta}_1\|^2_{H^4(\Gamma_s)} + \|\hat{\eta}_2\|^2_{H^2_0(\Gamma_s)}) \\ \leq \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^2_{\mathbf{H}} + \left\|\mathcal{A}_{1,1}(t)(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T\right\|^2_{\mathbf{H}} \\ \leq C_2(\|\Pi\mathbf{u}\|^2_{\mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0})} + \|\hat{\eta}_1\|^2_{H^4(\Gamma_s)} + \|\hat{\eta}_2\|^2_{H^2_0(\Gamma_s)}). \end{aligned}$$

The plan of the proof is the following:

- *Step 1:* We prove the existence of $\lambda_1 \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda \geq \lambda_1\} \subset \rho(\mathcal{A}_{1,1}(t))$ for all $t \geq 0$.
- *Step 2:* We prove the existence of $M_1 > 0$ such that $\mathcal{A}_{1,1} : [0, T] \to \mathfrak{A}_{\lambda_1, M_1}(\mathbf{H})$.
- *Step 3:* We prove that there exists $\kappa \in (0, 1)$ such that

  $$\mathcal{A}_{1,2} + \mathcal{A}_2 : [0, T] \to \mathcal{L}([\mathcal{D}(\mathcal{A}_{1,1}(t)), \mathbf{H}]_{1-\kappa}, \mathbf{H}),$$

  and

  $$\sup_{t \geq 0} \|\mathcal{A}_{1,2}(t) + \mathcal{A}_2(t)\|_{\mathcal{L}([\mathcal{D}(\mathcal{A}_{1,1}(t)), \mathbf{H}]_{1-\kappa}, \mathbf{H})} < +\infty. \tag{4.6.3}$$

  We then deduce, using Theorem 4.3.3, the existence of $\lambda_2 \in \mathbb{R}$ and $M_2 > 0$ such that $\mathcal{A} : [0, T] \to \mathfrak{A}_{\lambda_2, M_2}(\mathbf{H})$.
- *Step 4:* We use the previous splitting of $\mathcal{A}(t)$ to prove the uniform-in-time equivalence between the graph norm of $\mathcal{D}(\mathcal{A}(t))$ and $\|\cdot\|_{\mathbf{D}}$.

Step 1: Let $\lambda_b \in \mathbb{R}$ be such that $\{\lambda \in \mathbb{C} \mid \text{Re}\,\lambda \geq \lambda_b\} \subset \rho(\mathcal{A}_b)$ and suppose that $\lambda_1 \geq \max(\lambda_b, \lambda_0)$, where λ_0 is given by (4.4.2). Consider the system, for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geq \lambda_1$,

$$
\begin{aligned}
&\lambda \mathbf{u} + D_\pi \mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = F_1 \text{ in } \Omega_{\pi,0},\\
&\operatorname{div}\, \mathbf{u} = A_3\hat{\eta}_1 \text{ in } \Omega_{\pi,0},\\
&\mathbf{u} = \hat{\eta}_2\mathbf{e}_2 \text{ on } \Gamma_{\pi,0},\\
&\mathbf{u} = 0 \text{ on } \Gamma_i,\\
&u_2 = 0 \text{ and } p = 0 \text{ on } \Gamma_o,\\
&\mathbf{u} = 0 \text{ on } \Gamma_b,\\
&\lambda\hat{\eta}_1 - \hat{\eta}_2 = F_2 \text{ on } \Gamma_s,\\
&\lambda\hat{\eta}_2 - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} = F_3 \text{ in } \Gamma_s,\\
&\hat{\eta}_1 = 0 \text{ and } \hat{\eta}_{1,x} = 0 \text{ on } \{0, L\},
\end{aligned}
\tag{4.6.4}
$$

with $(F_1, F_2, F_3) \in \mathbf{H}$. The beam equation of this system is decoupled from the fluid equations. Hence we can solve (4.6.4) in two steps. First, as $\lambda \in \rho(\mathcal{A}_b)$, there exists a unique solution $(\hat{\eta}_1, \hat{\eta}_2) \in \big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\big) \times H_0^2(\Gamma_s)$ to the beam equation with the estimate

$$\|\hat{\eta}_1\|_{H^4(\Gamma_s)} + \|\hat{\eta}_2\|_{H_0^2(\Gamma_s)} \leq C\left(\|F_2\|_{H_0^2(\Gamma_s)} + \|F_3\|_{L^2(\Gamma_s)}\right).$$

Then, using that $A_3\hat{\eta}_1 = \operatorname{div}(-\tilde{\eta}_1\overline{u}_{\pi,1}\mathbf{e}_1 + z\tilde{\eta}_{1,x}\overline{u}_{\pi,1}\mathbf{e}_2)\hat{\eta}_1 \in H^1_{\text{lift}}(\Omega_{\pi,0})$ and $\hat{\eta}_2 \in H_0^2(\Gamma_s)$, Theorem 4.4.1 implies the existence of a unique $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ satisfying the fluid equation in (4.6.4), with the estimate

$$
\begin{aligned}
\|\mathbf{u}\|_{\mathbf{H}^2(\Omega_{\pi,0})} + \|p\|_{H^1(\Omega_{\pi,0})} &\leq C(\|F_1\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})} + \|A_3\hat{\eta}_1\|_{H^1(\Omega_{\pi,0})} + \|\hat{\eta}_2\|_{H_0^2(\Gamma_s)})\\
&\leq C\,\|(F_1, F_2, F_3)\|_{\mathbf{H}}.
\end{aligned}
$$

Finally remark that (4.6.4) is equivalent to

$$
\begin{cases}
\lambda \begin{pmatrix} \Pi\mathbf{u} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} = \mathcal{A}_{1,1}(t) \begin{pmatrix} \Pi\mathbf{u} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}\\
(I - \Pi)\mathbf{u} = \nabla N_s(\hat{\eta}_2) + \nabla N_{\text{div}}(A_3\hat{\eta}_1),\\
p = -\lambda N_{\text{div}}(A_3\hat{\eta}_1) - \lambda N_s(\hat{\eta}_2) + N_v(\mathbf{u}).
\end{cases}
\tag{4.6.5}
$$

Hence $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \lambda_1\} \subset \rho(\mathcal{A}_{1,1}(t))$ for all $t \geq 0$.

Step 2: For all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geq \lambda_1$, from the construction in Step 1 it can be seen that the resolvent of $\mathcal{A}_{1,1}(t)$ is given by the following formula

$$(\lambda I - \mathcal{A}_{1,1}(t))^{-1} = \begin{pmatrix} (\lambda I - A(t))^{-1} & \mathbf{T}(\lambda) \\ 0 & (\lambda I - \mathcal{A}_b)^{-1} \end{pmatrix},$$

with

$$\mathbf{T}(\lambda) = \begin{pmatrix} (\lambda I - A(t))^{-1}(\lambda_0 I - A(t))\Pi L_1(A_3\cdot, 0) \\ (\lambda I - A(t))^{-1}(\lambda_0 I - A(t))\Pi L_1(0, \cdot) \end{pmatrix}^T (\lambda I - \mathcal{A}_b)^{-1}.$$

There exists $M > 0$ such that, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geq \lambda_1$,

$$\left\|(\lambda I - A(t))^{-1}\right\|_{\mathcal{L}(\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}))} + \left\|(\lambda I - \mathcal{A}_b)^{-1}\right\|_{\mathcal{L}(H_b)} \leq \frac{M}{1+|\lambda|}.$$

It remains to estimate the term $\mathbf{T}(\lambda)$. Remark that

$$(\lambda I - A(t))^{-1}(\lambda_0 I - A(t))\Pi L_1 = (\lambda_0 - \lambda)(\lambda I - A(t))^{-1}\Pi L_1 + \Pi L_1.$$

Moreover, the following estimates hold, for all $\Phi \in H_b$,

$$\left\|\Pi L_1(((\lambda I - \mathcal{A}_b)^{-1}\Phi)_2)\right\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})} \leq \frac{C_{L_1}M}{1+|\lambda|}\left\|\Phi\right\|_{H_b},$$

where C_{L_1} is the continuity constant of L_1. Additionally

$$\begin{aligned}&\left\|(\lambda_0 - \lambda)(\lambda I - A(t))^{-1}\Pi L_1(((\lambda I - \mathcal{A}_b)^{-1}\Phi)_2)\right\|_{\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})}\\ &\leq |\lambda_0 - \lambda|\frac{C_{L_1}M^2}{(1+|\lambda|)^2}\left\|\Phi\right\|_{H_b} \leq \frac{M'}{1+|\lambda|}\left\|\Phi\right\|_{H_b},\end{aligned}$$

with $M' > 0$. Combining all the estimates we have proved that there exists a constant $M_1 > 0$ such that $\mathcal{A}_{1,1} : [0,T] \to \mathfrak{A}_{\lambda_1,M_1}(\mathbf{H})$.

Step 3: The idea is to prove that all the operators involved in $\mathcal{A}_{1,2}(t)$ and $\mathcal{A}_2(t)$ are of lower order compared to $\mathcal{A}_{1,1}(t)$. In order to described the regularity of the perturbation we use the constant $\kappa \in (0,1)$. Instead of using various $\kappa_1, \kappa_2, \ldots$ for all the operators, we keep the general notation κ, taking at the end κ sufficiently close to 1.

We first consider $\mathcal{A}_{1,2}(t)$, that is,

$$\mathcal{A}_{1,2}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & K_s A_{\alpha,\beta} & -(I + \overline{N}_s(\cdot))^{-1}\overline{N}_{\mathrm{div}}(A_3\cdot) + K_s\gamma\Delta \end{pmatrix},$$

with $K_s = (I + \overline{N}_s(\cdot))^{-1} - I$. We already know that $(I + \overline{N}_s)$ is an automorphism of $L^2(\Gamma_s)$ and $\overline{N}_{\mathrm{div}}(A_3\cdot) \in \mathcal{L}(H^2_0(\Gamma_s), H^{5/2}(\Gamma_s))$. Hence $(I + \overline{N}_s(\cdot))^{-1}\overline{N}_{\mathrm{div}}(A_3\cdot)$ can also be defined from $H^{2(\kappa+1)}(\Gamma_s) \cap H^2_0(\Gamma_s)$ with value in $L^2(\Gamma_s)$ for all $\kappa \in (0,1)$. For the terms $K_s A_{\alpha,\beta}$ and $K_s\gamma\Delta$ we refer to [56, Lemma 3.9] where it is proved, using a Fourier decomposition of the operator, that $K_s A_{\alpha,\beta} \in \mathcal{L}(H^{2(\kappa+1)}(\Gamma_s) \cap H^2_0(\Gamma_s), L^2(\Gamma_s))$ and $K_s\gamma\Delta \in \mathcal{L}(H^{2\kappa}_0(\Gamma_s), L^2(\Gamma_s))$ for $\frac{3}{4} < \kappa < 1$.

For the second part, we summarize the regularity of the operators involved in $\mathcal{A}_2(t)$:

Operators	Domain	Codomain
$\Pi A_{1,1}$	$H_0^2(\Gamma_s)$	$\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$
ΠA_2	$L^2(\Gamma_s)$	$\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$
B_∇	$\mathbf{H}^{2\kappa}(\Omega_{\pi,0})$, $1/2 < \kappa < 1$	$L^2(\Gamma_s)$
$A_{4,1}$	$H_0^2(\Gamma_s)$	$L^2(\Gamma_s)$
A_5	$H_0^2(\Gamma_s)$	$L^2(\Gamma_s)$
$B_\nabla(\nabla N_{\text{div}}(A_3\cdot))$	$H_0^2(\Gamma_s)$	$L^2(\Gamma_s)$
$\overline{N}_p(A_{1,1}\cdot)$	$H_0^2(\Gamma_s)$	$L^2(\Gamma_s)$
$B_\nabla(\nabla N_s(\cdot))$	$H_0^{2\kappa}(\Gamma_s)$, $3/4 < \kappa < 1$	$L^2(\Gamma_s)$
$\overline{N}_p(A_2\cdot)$	$L^2(\Gamma_s)$	$L^2(\Gamma_s)$

The terms involving $\overline{N}_v$ cannot be estimated separately. If we apply $\mathcal{A}_2(t)$ to $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(\mathcal{A}_{1,1}(t))$ the term with $\overline{N}_v$ is

$$\overline{N}_v[\Pi\mathbf{u} + \nabla N_{\text{div}}(A_3\hat{\eta}_1) + \nabla N_s(\hat{\eta}_2)]. \tag{4.6.6}$$

To estimate this term we use Lemma 4.5.2 and find $\frac{7}{8} < \kappa < 1$ such that (4.6.6) belongs to $L^2(\Gamma_s)$ for $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in [\mathcal{D}(\mathcal{A}_{1,1}(t)), \mathbf{H}]_{1-\kappa}$.

Finally we have proved the existence of $\kappa \in (\frac{7}{8}, 1)$ such that $\mathcal{A}_{1,2} + \mathcal{A}_2 : [0,T] \to \mathcal{L}([\mathcal{D}(\mathcal{A}_{1,1}(t)), \mathbf{H}]_{1-\kappa}, \mathbf{H})$. All the estimates mentioned above can be made independent of t by using the L^∞-norm in time of the periodic solution and the uniform-in-time equivalence between the graph norm of $\mathcal{A}_{1,1}(t)$ and $\|\cdot\|_{\mathbf{D}}$. This yields (4.6.3). We can then use Theorem 4.3.3 to prove the existence of $\lambda_2 \in \mathbb{R}$ and $M_2 > 0$ such that $\mathcal{A} : [0,T] \to \mathfrak{A}_{\lambda_2, M_2}(\mathbf{H})$.

Step 4: The existence of $C_2 > 0$ independent of t such that

$$\|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^2_{\mathbf{H}} + \left\|\mathcal{A}(t)(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T\right\|^2_{\mathbf{H}} \leq C_2 \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^2_{\mathbf{D}},$$

for all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(\mathcal{A}(t))$ is obtained through direct estimates using the regularity of the periodic solution. For the reverse inequality we already know that there exists $C' > 0$ independent of t such that, for all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(\mathcal{A}(t))$,

$$C' \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^2_{\mathbf{D}} \leq \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^2_{\mathbf{H}} + \left\|\mathcal{A}_{1,1}(t)(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T\right\|^2_{\mathbf{H}}. \tag{4.6.7}$$

Using that $\mathcal{A}_{1,2} + \mathcal{A}_2 : [0,T] \to \mathcal{L}([\mathcal{D}(\mathcal{A}_{1,1}(t)), \mathbf{H}]_{1-\kappa}, \mathbf{H})$ (with uniform estimate (4.6.3) on the norm) and the interpolation inequality in [43, Corollary 1.2.7] we obtain

$$\begin{aligned}\left\|(\mathcal{A}_{1,2}(t) + \mathcal{A}_2(t))(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T\right\|^2_{\mathbf{H}} &\leq C \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^2_{[\mathcal{D}(\mathcal{A}_{1,1}(t)), \mathbf{H}]_{1-\kappa}} \\ &\leq C(\kappa) \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^{2(1-\kappa)}_{\mathbf{H}} \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|^{2\kappa}_{\mathcal{D}(\mathcal{A}_{1,1}(t))}.\end{aligned}$$

Using the young inequality with a parameter $\varepsilon > 0$,

$$\left\|(\mathcal{A}_{1,2}(t) + \mathcal{A}_2(t))(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T\right\|_{\mathbf{H}}^2 \leq C(\kappa)(1-\kappa)\left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\kappa}} \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|_{\mathbf{H}}^2 + C(\kappa)\kappa\varepsilon^\kappa \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|_{\mathcal{D}(\mathcal{A}_{1,1}(t))}^2 .$$

The rest of the calculations are standard. Taking $\varepsilon > 0$ small enough and using (4.6.7) we obtain a constant $C_1 > 0$ independent of t, such that

$$C_1 \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|_{\mathbf{D}}^2 \leq \|(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)\|_{\mathbf{H}}^2 + \left\|\mathcal{A}(t)(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T\right\|_{\mathbf{H}}^2 ,$$

which concludes the proof. □

We are now able to prove the existence of a parabolic evolution operator for the family $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{t\geq 0}$.

Theorem 4.6.2. There exists a unique parabolic evolution operator U for the family $(\mathcal{A}(t), \mathcal{D}(\mathcal{A}(t)))_{t\geq 0}$ with regularity subspace

$$E_\theta = \mathbf{V}^{2\theta}_{n,\Gamma_d} \times \left(H_0^{2(\theta+1)}(\Gamma_s) \cap H_0^2(\Gamma_s)\right) \times H_0^{2\theta}(\Gamma_s),$$

where θ is defined in (4.6.1).

Proof. We have to prove that $\mathcal{A}(t)$ satisfies (B_1)–(B_2). For all $t \geq 0$ consider the isomorphism

$$\Theta(t) : \begin{cases} \mathcal{D}(\mathcal{A}(t)) \to \mathcal{D}(A(t)) \times \left(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\right) \times H_0^2(\Gamma_s) \\ (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \mapsto (\Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2), \hat{\eta}_1, \hat{\eta}_2). \end{cases} \tag{4.6.8}$$

Using Lemma 4.8.4 the operator L_1 can be defined from $H^1(\Omega_{\pi,0}) \times L^2(\Gamma_s)$ with values in $\mathbf{L}^2(\Omega_{\pi,0})$. Hence $\Theta(t)$ is also an automorphism in $\mathbf{H}$. Using Lemma 4.8.2 we deduce that

$\Theta(t)$ is an isomorphism

$$[\mathcal{D}(\mathcal{A}(t)), \mathbf{H}]_{1-\theta} \to [\mathcal{D}(A(t)), \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta} \times \left(H^{2(\theta+1)}(\Gamma_s) \cap H_0^2(\Gamma_s)\right) \times H_0^{2\theta}(\Gamma_s).$$

For $\theta \in (0, \frac{1}{4})$, recalling that $\mathcal{D}(A(t)) = V \cap \mathbf{H}^2(\Omega_{\pi,0})$ and $[V \cap \mathbf{H}^2, \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1/2} = \mathcal{D}(A_s^{1/2}) = V$, Lemma 4.8.3 implies

$$[\mathcal{D}(A(t)), \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta} = [V \cap \mathbf{H}^2(\Omega_{\pi,0}), \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta} = \mathbf{V}^{2\theta}_{n,\Gamma_d}(\Omega_{\pi,0}).$$

Finally, for all $t \geq 0$,

$$[\mathcal{D}(\mathcal{A}(t)), \mathbf{H}]_{1-\theta} = \{(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathbf{V}^{2\theta}_{n,\Gamma_d}(\Omega_{\pi,0}) \times \left(H^{2(\theta+1)}(\Gamma_s) \cap H_0^2(\Gamma_s)\right) \times H_0^{2\theta}(\Gamma_s) \mid \Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) \in \mathbf{V}^{2\theta}_{n,\Gamma_d}(\Omega_{\pi,0})\},$$

is algebraically equal to E_θ as the compatibility condition does not impose conditions on the data. It remains to prove that the norm on $E_\theta(\mathcal{A}(t)) = [\mathcal{D}(\mathcal{A}(t)), \mathbf{H}]_{1-\theta}$ is uniformly-in-time equivalent to the norm of E_θ. For all $t \geq 0$ the map $\Theta(t)$ is described by the matrix

$$\Theta(t) = \begin{pmatrix} I & -\Pi L_1(A_3\cdot, 0) & -\Pi L_1(0, \cdot) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

and its inverse is

$$\Theta(t)^{-1} = \begin{pmatrix} I & \Pi L_1(A_3\cdot, 0) & \Pi L_1(0, \cdot) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Using Lemma 4.8.4 we deduce uniform-in-time estimates for the operator L_1 and there exists a constant $C_{L1} > 0$, independent of t, such that

$$\begin{aligned} \left\| \Theta(t)^{-1} (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T \right\|_{\mathcal{D}(\mathcal{A}(t))} &\leq C_2 \left\| \Theta(t)^{-1} (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T \right\|_{\mathbf{D}} \\ &\leq C_2 C_{L1} \left\| (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \right\|_{\mathbf{D}} \\ &\leq C_2 C_{L1} C_1 \left\| (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \right\|_{\mathcal{D}(\mathcal{A}(t))}, \end{aligned}$$

for all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathcal{D}(A(t)) \times \left(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\right) \times H_0^2(\Gamma_s)$, where $C_1 > 0$ and $C_2 > 0$ are the constants, independent of t, appearing in the equivalence between the graph norm of $\mathcal{D}(\mathcal{A}(t))$ and $\|\cdot\|_{\mathbf{D}}$. We also have $C_{L2} > 0$, independent of t, such that

$$\left\| \Theta(t)^{-1} (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T \right\|_{\mathbf{H}} \leq C_{L2} \left\| (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \right\|_{\mathbf{H}},$$

for all $(\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \in \mathbf{H}$. Using interpolation we obtain

$$\left\| \Theta(t)^{-1} \right\|_{\mathcal{L}(E_\theta, E_\theta(\mathcal{A}(t)))} \leq (C_2 C_{L1} C_1)^\theta C_{L2}^{1-\theta}.$$

Similar result can be proved on $\Theta(t)$. Finally,

$$\begin{aligned} \left\| (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)) \right\|_{E_\theta(\mathcal{A}(t))} &= \left\| \Theta(t)^{-1} \Theta(t) (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T \right\|_{E_\theta(\mathcal{A}(t))} \\ &\leq C \left\| \Theta(t) (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2)^T \right\|_{E_\theta} \\ &\leq C \left(\left\| (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \right\|_{E_\theta} + \left\| (-\Pi L_1(A_3 \hat{\eta}_1, \hat{\eta}_2), 0, 0) \right\|_{E_\theta} \right), \end{aligned}$$

and, using interpolation on L_1, Lemma 4.8.4 and uniform estimates on the periodic solution for the term A_3, we find $C_{L3} > 0$, independent of t, such that

$$\left\| (-\Pi L_1(A_3 \hat{\eta}_1, \hat{\eta}_2), 0, 0) \right\|_{E_\theta} \leq C_{L3} \left\| (\Pi\mathbf{u}, \hat{\eta}_1, \hat{\eta}_2) \right\|_{E_\theta}.$$

The other inequality in the equivalence between the norm of $E_\theta(\mathcal{A}(t))$ and E_θ is similar. Hence $\mathcal{A}(t)$ satisfies (B_1).

It remains to prove (B_2). We introduce the operator $(\mathring{\mathcal{A}}(t), \mathcal{D}(\mathring{\mathcal{A}}(t)))_{t\geq 0}$ in $\mathbf{H}$ defined by

$$\mathcal{D}(\mathring{\mathcal{A}}(t)) = \mathcal{D}(A(t)) \times \Big(H^4(\Gamma_s) \cap H_0^2(\Gamma_s)\Big) \times H_0^2(\Gamma_s),$$

and the formula $\mathcal{A}(t) = \mathring{\mathcal{A}}(t) \circ \Theta(t)$. Remark that $\mathcal{D}(\mathring{\mathcal{A}}(t))$ does not depend on t since $\mathcal{D}(A(t)) = V \cap \mathbf{H}^2(\Omega_{\pi,0})$. Setting, for an operator B, $B_\lambda = B - \lambda I$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(\mathcal{A}(t)) \arrow[r, "\Theta(t)"] \arrow[d, "\mathcal{A}_{\lambda_2}(t)"'] & \mathcal{D}(\mathring{\mathcal{A}}(t)) \arrow[d, "\mathring{\mathcal{A}}_{\lambda_2}(t)"] \\
\mathbf{H} \arrow[r, "I_{\mathbf{H}}"'] & \mathbf{H}
\end{tikzcd}$$

We want to prove that $\mathcal{A}_{\lambda_2}^{-1} \in \mathcal{C}^\rho([0,T]; \mathcal{L}(\mathbf{H}, E_\theta))$. As the domain of $\mathcal{A}(t)$ depends on time we cannot directly utilise the Hölder regularity of $\mathcal{A}(t)$. To get around this issue we show that $\mathring{\mathcal{A}} \in \mathcal{C}^\rho([0,T]; \mathrm{Iso}(\mathcal{D}(\mathring{\mathcal{A}}(t)), \mathbf{H}))$ and that $\Theta^{-1} \in \mathcal{C}^\rho([0,T]; \mathcal{L}(\mathcal{D}(\mathring{\mathcal{A}}(t), E_\theta))$. Finally we conclude using that $\mathcal{A}_{\lambda_2}^{-1} = \Theta^{-1} \circ (\mathring{\mathcal{A}}_{\lambda_2})^{-1}$.

The Hölder regularity of the operator $\mathring{\mathcal{A}}(t)$ follows directly from the regularity of the periodic solution and standard properties of compositions, products and sums of Hölder functions. The Hölder regularity of Θ follows from Lemma 4.8.4 and the previous uniform-in-time equivalence between the norm of $E_\theta(\mathcal{A}(t))$ and E_θ.

Hence $\mathcal{A}(t)$ satisfies (B_2) and Theorem 4.3.2 yields the existence of a parabolic evolution operator with regularity subspace E_θ. $\square$

Remark 4.6.1. Consider the Cauchy problem

$$\begin{cases} \mathbf{y}'(t) = \mathcal{A}(t)\mathbf{y}(t), \text{ for } T \geq t > 0, \\ \mathbf{y}(0) = x, \end{cases} \tag{4.6.9}$$

Using [4, Theorem 2.5.1, Chapter IV] we know that, for $x \in \mathbf{H}$, (4.6.9) has a unique solution $\mathbf{y} \in \mathcal{C}([0,T]; \mathbf{H}) \cap \mathcal{C}^1(]0,T]; \mathbf{H})$ and, for all $t \in (0,T]$, $\mathbf{y}(t) \in \mathcal{D}(\mathcal{A}(t))$. In particular $\mathcal{A}(t)\mathbf{y}(t) \in \mathcal{C}((0,T]; \mathbf{H})$ and the uniform-in-time equivalence of the graph norm on $\mathcal{D}(\mathcal{A}(t))$ and of $\|\cdot\|_{\mathbf{D}}$ gives $\mathbf{y} \in \mathcal{C}((0,T]; \mathbf{D})$.

4.7 Stabilization of the linear system

In this section we study the stabilisation of the linear system

$$
\begin{aligned}
&\mathbf{v}_t - \nu\Delta\mathbf{v} + D_\pi(t)\mathbf{v} - A_{1,1}\hat{\eta}_1 - A_2\hat{\eta}_2 + \nabla q - \omega\mathbf{v} = 0 \ \text{ in } Q_\infty^{\pi,0},\\
&\operatorname{div}\mathbf{v} = A_3\hat{\eta}_1 \ \text{ in } Q_\infty^{\pi,0},\\
&\mathbf{v} = \hat{\eta}_2\mathbf{e}_2 \ \text{ on } \Sigma_\infty^{\pi,0},\\
&\mathbf{v} = \mathbf{v}_c \ \text{ on } \Sigma_\infty^{i},\\
&v_2 = 0 \ \text{ and } \ q = 0, \ \text{ on } \Sigma_\infty^{o},\\
&\mathbf{v} = 0 \ \text{ on } \Sigma_\infty^{b}, \ \ \mathbf{v}(0) = \mathbf{v}^0 \ \text{ in } \Omega_{\pi,0},\\
&\hat{\eta}_{1,t} - \omega\hat{\eta}_1 - \hat{\eta}_2 = 0 \text{ on } \Sigma_\infty^{s},\\
&\hat{\eta}_{2,t} - \omega\hat{\eta}_2 - \beta\hat{\eta}_{1,xx} - \gamma\hat{\eta}_{2,xx} + \alpha\hat{\eta}_{1,xxxx} - A_{4,1}\hat{\eta}_1\\
&\qquad = -J_{\eta^0_{\pi,1}}\mathbf{e}_2\cdot\sigma(\mathbf{v},q)_{|\Gamma_{\eta^0_{\pi,1}}}\mathbf{n}_{\eta^0_{\pi,1}} \ \text{ in } \Sigma_\infty^{s},\\
&\hat{\eta}_1 = 0 \ \text{ and } \ \hat{\eta}_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\hat{\eta}_1(0) = \hat{\eta}_1^0 \ \text{ and } \ \hat{\eta}_2(0) = \hat{\eta}_2^0 \ \text{ in } \Gamma_s.
\end{aligned}
\tag{4.7.1}
$$

The objective is to stabilize (4.7.1) with a control $\mathbf{v}_c$ acting on an open interval $\Gamma_c \subset \Gamma_i$. System (4.7.1) can be written as a matrix evolution equation. We are interested in the stabilisation of classical solutions for this problem. For parabolic evolution equations, we refer to [40] for the definitions of solutions on unbounded intervals, and to [4] for the general definitions when the underlying operator has a non-constant domain. By extension, the term 'classical solutions' is also used for (4.7.1). Typically, for classical solutions of (4.7.1), the velocity of the fluid $\mathbf{v}$ belongs to $\mathcal{C}^0([0,+\infty);\mathbf{L}^2(\Omega_{\pi,0})) \cap \mathcal{C}^0((0,+\infty);\mathbf{H}^2(\Omega_{\pi,0})) \cap \mathcal{C}^1((0,+\infty);\mathbf{L}^2(\Omega_{\pi,0}))$. The initial data $(\Pi\mathbf{v}^0, \hat{\eta}_1^0, \hat{\eta}_2^0)$ in the matrix formulation are chosen in $\mathbf{H}$. To ensure the continuity of $\mathbf{v}$ at time $t = 0$ we introduce the space

$$
\mathbf{H}_{cc} := \{(\mathbf{v}^0, \hat{\eta}_1^0, \hat{\eta}_2^0) \in \hat{\mathbf{H}} \mid \mathbf{v}^0 - L_1(0)(A_3(0)\hat{\eta}_1^0, \hat{\eta}_2^0) \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})\},
\tag{4.7.2}
$$

where $L_1(0)$ is the operator L_1 computed at time $t = 0$ i.e. with $\overline{\mathbf{u}}_\pi(0)$.

Proposition 4.7.1. A pair $(\mathbf{v}, q)$ such that

- $\mathbf{v} \in \mathcal{C}^0([0,+\infty);\mathbf{L}^2(\Omega_{\pi,0})) \cap \mathcal{C}^0((0,+\infty);\mathbf{H}^2(\Omega_{\pi,0})) \cap \mathcal{C}^1((0,+\infty);\mathbf{L}^2(\Omega_{\pi,0}))$,
- $q \in \mathcal{C}^0((0,+\infty);H^1(\Omega_{\pi,0}))$,

satisfies $(4.7.1)_1$–$(4.7.1)_6$ if and only if

(4.7.3)

$$
\begin{aligned}
&\Pi\mathbf{v}_t = (A(t) + \omega I)\Pi\mathbf{v} + (\lambda_0 I - A(t))\Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) + (\lambda_0 I - A(t))\Pi L_{\Gamma_i}(\mathbf{v}_c),\\
&(I - \Pi)\mathbf{v} = \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1) + \nabla N_i(\mathbf{v}_c),\\
&q = -N_{\mathrm{div}}(A_3(\hat{\eta}_{1,t} - \hat{\eta}_1)) - N_{\mathrm{div}}(A_{3,t}\hat{\eta}_1) - N_s(\hat{\eta}_{2,t} - \omega\hat{\eta}_2) - N_{\Gamma_i}(\mathbf{v}_{c,t} - \omega\mathbf{v}_c) + N_v(\Pi\mathbf{v})\\
&\ + N_v(\nabla N_{\mathrm{div}}(A_3\hat{\eta}_1) + \nabla N_s(\hat{\eta}_2)) + N_v(\nabla N_i(\mathbf{v}_c)).
\end{aligned}
$$

Proof. The proof is similar to Theorem 4.4.4. □

We are going to reformulate (4.7.1) as a matrix evolution equation. The regularity of the operator N_v is increased in Lemma 4.5.2 when the operator is applied to specific functions. When $\mathbf{w} \in \mathbf{H}^2(\Omega_{\pi,0})$, using the definition (4.4.12) of N_v and a reverse of the integration by part that led to (4.4.11), the function $N_v(\mathbf{w}) \in L^2(\Omega_{\pi,0})$ can be defined as a continuous linear form on $L^2(\Omega_{\pi,0})$ endowed with the $(H^1(\Omega_{\pi,0}))'$-topology. Using the Hahn-Banach theorem, this form can be extended as a continuous form on $(H^1(\Omega_{\pi,0}))'$. The reflexivity of $H^1(\Omega_{\pi,0})$ and the definition of the duality with $L^2(\Omega_{\pi,0})$ as a pivot space then show that $N_v(\mathbf{w}) \in (H^1(\Omega_{\pi,0}))'' = H^1(\Omega_{\pi,0})$. Hence $N_v(\nabla N_i(\mathbf{v}_c)) \in H^1(\Omega_{\pi,0})$ and its trace on $\Gamma_{\pi,0}$ is well defined. This is required as this term is treated separately in the coming formulation (4.7.4).

For classical solutions, system (4.7.1) is equivalent to

(4.7.4)
$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \Pi\mathbf{v} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} = (\mathcal{A}(t) + \omega I)\begin{pmatrix} \Pi\mathbf{v} \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} + \mathcal{B}_c\mathbf{v}_c + \mathcal{B}_d\mathbf{v}_{c,t}, \quad \begin{pmatrix} \Pi\mathbf{v}(0) \\ \hat{\eta}_1(0) \\ \hat{\eta}_2(0) \end{pmatrix} = \begin{pmatrix} \Pi\mathbf{v}^0 \\ \hat{\eta}_1^0 \\ \hat{\eta}_1^0 \end{pmatrix}, \\ (I - \Pi)\mathbf{v} = \nabla N_s(\hat{\eta}_2) + \nabla N_{\mathrm{div}}(A_3\hat{\eta}_1) + \nabla N_i(\mathbf{u}_c), \\ q = -N_{\mathrm{div}}(A_3(\hat{\eta}_{1,t} - \hat{\eta}_1)) - N_{\mathrm{div}}(A_{3,t}\hat{\eta}_1) - N_s(\hat{\eta}_{2,t} - \omega\hat{\eta}_2) - N_{\Gamma_i}(\mathbf{v}_{c,t} - \omega\mathbf{v}_c) + N_v(\mathbf{v}). \end{cases}$$

with

$$\mathcal{B}_c\mathbf{v}_c = \begin{pmatrix} (\lambda_0 - A(t))\Pi L_{\Gamma_i}(\mathbf{v}_c) \\ 0 \\ \omega(I + \overline{N}_s)^{-1}[\overline{N}_{\Gamma_i}(\mathbf{v}_c) + \overline{N}_v(\nabla N_i(\mathbf{v}_c))] \end{pmatrix}$$
$$\mathcal{B}_d\mathbf{v}_{c,t} = \begin{pmatrix} 0 \\ 0 \\ -(I + \overline{N}_s)^{-1}N_i(\mathbf{v}_c) \end{pmatrix}.$$

In the previous formulation the time derivative of the control variable $\mathbf{v}_c$ appears. To fit into the standard control framework for evolution equation we adapt the techniques developed in [44]. Let us mention that the strategy used in [44] cannot be directly applied to our parabolic evolution equation since it involves non-constant operators. We split $\mathbf{v}_c$ into two control variables that will be considered as state variables: $\mathbf{v}_c = \mathbf{v}_c^1 + \mathbf{v}_c^2$. Consider $\mathbf{g}_1 = \mathbf{v}_{c,t}^1 - 2\Delta_z\mathbf{v}_c^1$ and $\mathbf{g}_2 = \mathbf{v}_{c,t}^2 - \Delta_z\mathbf{v}_c^2$. The pair $(\mathbf{g}_1, \mathbf{g}_2)$ is the new control variable. The idea to consider two heat equations with different diffusion coefficients is crucial to solve the continuation problem associated with this formulation. This idea to 'double' the control variable already appears in [44] in a discrete case. Remark that the heat equation is used here to ensure that the extended operator remains analytic (or, more precisely, that we can still construct a parabolic evolution operator). The idea of considering an extended system with a heat equation on the boundary was already used in [7] for the Navier–Stokes equations.

We introduce the space

$$\mathbf{H}_e = \mathbf{H} \times \mathbf{L}^2(\Gamma_i) \times \mathbf{L}^2(\Gamma_i).$$

The extended system, equivalent to (4.7.4) is

$$\text{(4.7.5)} \qquad \frac{d}{dt}\begin{pmatrix}\Pi\mathbf{v}\\ \hat{\eta}_1\\ \hat{\eta}_2\\ \mathbf{v}_c^1\\ \mathbf{v}_c^2\end{pmatrix} = \mathcal{A}_e(t)\begin{pmatrix}\Pi\mathbf{v}\\ \hat{\eta}_1\\ \hat{\eta}_2\\ \mathbf{v}_c^1\\ \mathbf{v}_c^2\end{pmatrix} + \mathcal{B}_e\begin{pmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{pmatrix}, \quad \begin{pmatrix}\Pi\mathbf{v}(0)\\ \hat{\eta}_1(0)\\ \hat{\eta}_2(0)\\ \mathbf{v}_c^1(0)\\ \mathbf{v}_c^2(0)\end{pmatrix} = \begin{pmatrix}\Pi\mathbf{v}^0\\ \hat{\eta}_1^0\\ \hat{\eta}_2^0\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}$$

where $(\mathcal{A}_e(t), \mathcal{D}(\mathcal{A}_e(t))_{t\geq 0}$ is the unbounded operator on $\mathbf{H}_e$ defined by

$$\text{(4.7.6)} \qquad \mathcal{A}_e(t) = \begin{pmatrix} \mathcal{A}(t) + \omega I & \begin{pmatrix}\mathcal{B}_c + 2\mathcal{B}_d\Delta_z & \mathcal{B}_c + \mathcal{B}_d\Delta_z\end{pmatrix} \\ 0 & \begin{pmatrix}2\Delta_z & 0\\ 0 & \Delta_z\end{pmatrix}\end{pmatrix},$$

(4.7.7)

$$\begin{aligned}\mathcal{D}(\mathcal{A}_e(t)) := \{(\Pi\mathbf{v}, \hat{\eta}_1, \hat{\eta}_2, \mathbf{v}_c^1, \mathbf{v}_c^2) \,:\, &\Pi\mathbf{v} \in \mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0}),\ \hat{\eta}_1 \in (H^4(\Gamma_s)\cap H^2_0(\Gamma_s)),\\ &\hat{\eta}_2 \in H^2_0(\Gamma_s),\ (\mathbf{v}_c^1, \mathbf{v}_c^2) \in \Big(\mathbf{H}^2(\Gamma_i)\cap\mathbf{H}^1_0(\Gamma_i)\Big)^2 \mid \\ &\Pi\mathbf{u} - \Pi L_1(A_3\hat{\eta}_1, \hat{\eta}_2) - \Pi L_{\Gamma_i}(\mathbf{v}_c^1 + \mathbf{v}_c^2) \in \mathcal{D}(A(t))\}\end{aligned}$$

and the control operator $\mathcal{B}_e \in \mathcal{L}(\mathbf{L}^2(\Gamma_i)\times\mathbf{L}^2(\Gamma_i), \mathbf{H}_e)$ is defined by

$$\text{(4.7.8)} \qquad \mathcal{B}_e\begin{pmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -(I+\overline{N}_s)^{-1}N_i(\mathbf{g}_1+\mathbf{g}_2)\\ \mathbf{g}_1\\ \mathbf{g}_2\end{pmatrix}.$$

Let us calculate the adjoint operators of $\mathcal{A}_e(t)$ and $\mathcal{B}_e$.

Proposition 4.7.2. The adjoint of $(\mathcal{A}_e(t), \mathcal{D}(\mathcal{A}_e(t)))_{t\geq 0}$ is defined by

$$\text{(4.7.9)} \qquad \begin{aligned}&\mathcal{D}(\mathcal{A}_e^*(t)) := \{(\Pi\mathbf{w}, k_1, k_2, \mathbf{w}_c^1, \mathbf{w}_c^2) \in \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}) \times (H^4(\Gamma_s)\cap H^2_0(\Gamma_s))\\ &\times H^2_0(\Gamma_s) \times \Big(\mathbf{H}^2(\Gamma_i)\cap\mathbf{H}^1_0(\Gamma_i)\Big)^2 \mid \Pi\mathbf{w} - \Pi D_1((I+\overline{N}_s)^{-1})k_2) \in \mathcal{D}(A^*(t))\}.\end{aligned}$$

and

$$\text{(4.7.10)} \qquad \mathcal{A}_e^*(t) = \begin{pmatrix}\mathcal{A}^*(t) + \omega I & 0\\ \Big(\mathcal{B}_c + 2\mathcal{B}_d\Delta_z \quad \mathcal{B}_c + \mathcal{B}_d\Delta_z\Big)^* & \begin{pmatrix}2\Delta_z & 0\\ 0 & \Delta_z\end{pmatrix}\end{pmatrix},$$

where $\Big(\mathcal{B}_c + 2\mathcal{B}_d\Delta_z \quad \mathcal{B}_c + \mathcal{B}_d\Delta_z\Big)^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}^*(t)), \mathbf{L}^2(\Gamma_i)\times\mathbf{L}^2(\Gamma_i))$ is defined by

$$\Big(\mathcal{B}_c + 2\mathcal{B}_d\Delta_z \quad \mathcal{B}_c + \mathcal{B}_d\Delta_z\Big)^*\begin{pmatrix}\Pi\mathbf{w}\\ k_1\\ k_2\end{pmatrix} = \begin{pmatrix}-\sigma(\mathbf{w}, q)\,\mathbf{n}\\ -\sigma(\mathbf{w}, q)\,\mathbf{n}\end{pmatrix}$$

for $(\Pi\mathbf{w}, k_1, k_2) \in \mathcal{D}(\mathcal{A}^*(t))$ and $\mathbf{w} = \Pi\mathbf{w} + \nabla N_s(k_2)$, $q = -N_s(k_2) + N_{v^*}(\mathbf{v})$.

Proof. The proof is done using the same techniques, based on integration-by-parts, as in the proofs of Lemma 4.5.4 and Theorem 4.5.2. $\square$

Proposition 4.7.3. The adjoint of $\mathcal{B}_e$ is the operator $\mathcal{B}_e^* \in \mathcal{L}(\mathbf{H}_e, \mathbf{L}^2(\Gamma_i) \times \mathbf{L}^2(\Gamma_i))$ defined by

$$\mathcal{B}_e^* \begin{pmatrix} \Pi\mathbf{w} \\ k_1 \\ k_2 \\ \mathbf{w}_c^1 \\ \mathbf{w}_c^2 \end{pmatrix} = \begin{pmatrix} \mathbf{w}_c^1 - N_s((I+\overline{N}_s)^{-1}k_2)_{|\Gamma_i}\mathbf{n} \\ \mathbf{w}_c^2 - N_s((I+\overline{N}_s)^{-1}k_2)_{|\Gamma_i}\mathbf{n} \end{pmatrix}.$$

Proof. Consider $(\Pi\mathbf{w}, k_1, k_2, \mathbf{w}_c^1, \mathbf{w}_c^2) \in \mathbf{H}_e$ and $(\mathbf{g}_1, \mathbf{g}_2) \in \Big(\mathbf{L}^2(\Gamma_i)\Big)^2$. By definition of $\mathcal{B}_e$,

$$\left\langle \mathcal{B}_e \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix}, \begin{pmatrix} \Pi\mathbf{w} \\ k_1 \\ k_2 \\ \mathbf{w}_c^1 \\ \mathbf{w}_c^2 \end{pmatrix} \right\rangle_{\mathbf{H}_e} = -\int_{\Gamma_s} [(I+\overline{N}_s)^{-1}N_i(\mathbf{g}_1+\mathbf{g}_2)]k_2 + \int_{\Gamma_i} \mathbf{g}_1 \cdot \mathbf{w}_c^1 + \int_{\Gamma_i} \mathbf{g}_2 \cdot \mathbf{w}_c^2.$$

The integral on Γ_s becomes

$$\int_{\Gamma_s} N_i(\mathbf{g}_1+\mathbf{g}_2)(I+\overline{N}_s)^{-1}k_2 = \int_{\Gamma_s} N_i(\mathbf{g}_1+\mathbf{g}_2)\frac{\partial N_s}{\partial \mathbf{n}}((I+\overline{N}_s)^{-1}k_2),$$

and using that

$$\begin{aligned} 0 &= \int_{\Omega_{\pi,0}} \Delta N_s((I+\overline{N}_s)^{-1}k_2)N_i(\mathbf{g}_1+\mathbf{g}_2) \\ &= \int_{\Gamma_s} N_i(\mathbf{g}_1+\mathbf{g}_2)\frac{\partial N_s}{\partial \mathbf{n}}((I+\overline{N}_s)^{-1}k_2) - \int_{\Gamma_i} (\mathbf{g}_1+\mathbf{g}_2)\cdot N_s((I+\overline{N}_s)^{-1}k_2)\mathbf{n} \\ &+ \int_{\Omega_{\pi,0}} \Delta N_i(\mathbf{g}_1+\mathbf{g}_2)N_s((I+\overline{N}_s)^{-1}k_2), \end{aligned}$$

we obtain the desired result. $\square$

Theorem 4.7.1. For all $(\Pi\mathbf{v}^0, \hat{\eta}_1^0, \hat{\eta}_2^0, \mathbf{v}_{c,0}^1, \mathbf{v}_{c,0}^2) \in \mathbf{H}_e$ there exists

$$(\mathbf{g}_1, \mathbf{g}_2) \in \Big(\mathcal{C}_0^\rho([0,+\infty); \mathbf{L}^2(\Gamma_i))\Big)^2,$$

such that the classical solution $(\Pi\mathbf{v}, \hat{\eta}_1, \hat{\eta}_2, \mathbf{v}_c^1, \mathbf{v}_c^2)$ to (4.7.5), which belongs to

- $\Pi\mathbf{v} \in \mathcal{C}^0([0,+\infty); \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})) \cap \mathcal{C}^0((0,+\infty); \mathbf{V}^2_{n,\Gamma_d}(\Omega_{\pi,0})) \cap \mathcal{C}^1((0,+\infty); \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}))$,
- $\hat{\eta}_1 \in \mathcal{C}^0([0,+\infty); H_0^2(\Gamma_s)) \cap \mathcal{C}^0((0,+\infty); H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \cap \mathcal{C}^1((0,+\infty); H_0^2(\Gamma_s))$,

- $\hat{\eta}_2 \in \mathcal{C}^0([0,+\infty); L^2(\Gamma_s)) \cap \mathcal{C}^0((0,+\infty); H^2_0(\Gamma_s)) \cap \mathcal{C}^1((0,+\infty); L^2(\Gamma_s))$,
- $\mathbf{v}^1_c \in \mathcal{C}^0([0,+\infty); \mathbf{L}^2(\Gamma_i)) \cap \mathcal{C}^0((0,+\infty); \mathbf{H}^2(\Gamma_i) \cap \mathbf{H}^1_0(\Gamma_s)) \cap \mathcal{C}^1((0,+\infty); \mathbf{L}^2(\Gamma_i))$,
- $\mathbf{v}^2_c \in \mathcal{C}^0([0,+\infty); \mathbf{L}^2(\Gamma_i)) \cap \mathcal{C}^0((0,+\infty); \mathbf{H}^2(\Gamma_i) \cap \mathbf{H}^1_0(\Gamma_s)) \cap \mathcal{C}^1((0,+\infty); \mathbf{L}^2(\Gamma_i))$,

satisfies, for all $a > 0$,

$$\sup_{t>a} \left\| (\Pi\mathbf{v}(t), \hat{\eta}_1(t), \hat{\eta}_2(t), \mathbf{v}^1_c(t), \mathbf{v}^2_c(t)) \right\|_{\mathbf{D}_e} < +\infty,$$

with $\mathbf{D}_e = \mathbf{D} \times [\mathbf{H}^2(\Gamma_i) \cap \mathbf{H}^1_0(\Gamma_i)]^2$ (recall that $\mathbf{D}$ and its norm are defined in (4.6.2)).

Proof. The matrix describing the unbounded operator $\mathcal{A}_e(t)$ is triangular. Using similar arguments as in the proofs of Theorems 4.6.1, 4.6.2 and the analyticity of the heat equation, we can prove the existence of an evolution operator U_e for the family $(\mathcal{A}_e(t), \mathcal{D}(\mathcal{A}_e(t)))_{t\geq 0}$ with regularity subspace $E_{\theta,e} = E_\theta \times [\mathbf{H}^{2\theta}(\Gamma_i)]^2$, where we recall that $0 < \theta < \frac{1}{4}$ is defined by (4.6.1). The Rellich theorem shows that $E_{\theta,e}$ is compactly embedded in $\mathbf{H}_e$. Moreover, using the same reasoning as in Theorem 4.6.1, the graph norm of $\mathcal{D}(\mathcal{A}_e(t))$ is uniformly-in-time equivalent to the $\mathbf{D}_e$-norm. We can therefore use the results presented in Section 4.3. Let us prove the condition (4.3.10) from Remark 4.3.2. Let $\mathbf{y} = (\Pi\mathbf{w}, k_1, k_2, \mathbf{w}^1_c, \mathbf{w}^2_c)$ be a classical solution to

$$-\mathbf{y}'(t) = \mathcal{A}^*_e(t)\mathbf{y}(t) \text{ in } [0,+\infty).$$

Owing to Proposition 4.7.2, this solution satisfies the following system

$$\begin{aligned}
& -\mathbf{w}_t - \nu\Delta\mathbf{w} + D^a_\pi(t)\mathbf{w} + \nabla q - \omega\mathbf{w} = 0 \text{ in } Q^{\pi,0}_\infty,\\
& \operatorname{div} \mathbf{w} = 0 \text{ in } Q^{\pi,0}_\infty,\\
& \mathbf{w} = k_2\mathbf{e}_2 \text{ on } \Sigma^{\pi,0}_\infty,\\
& \mathbf{w} = 0 \text{ on } \Sigma^i_\infty,\\
& w_2 = 0 \text{ and } q = (\overline{\mathbf{u}}_\pi(t)\cdot\mathbf{n})w_1 \text{ on } \Sigma^o_\infty,\\
& \mathbf{w} = 0 \text{ on } \Sigma^b_\infty,\\
& -k_{1,t} + k_2 - (-A_{\alpha,\beta})^{-1}A^*_{1,1}\mathbf{w} - (-A_{\alpha,\beta})^{-1}(-A^*_3 + A^*_{4,1} + A^*_5)k_2\\
& \qquad + (-A_{\alpha,\beta})^{-1}A^*_3 q - \omega k_1 = 0 \text{ on } \Sigma^s_\infty,\\
& -k_{2,t} + (\eta_\pi + C_\pi(t))k_2 + \beta k_{1,xx} - \gamma k_{2,xx} - \alpha k_{1,xxxx} - A^*_2\mathbf{w}\\
& \qquad + J_{\eta^0_{\pi,1}}\mathbf{e}_2\cdot\sigma(\mathbf{w},q)_{|\Gamma_{\eta^0_{\pi,1}}}\mathbf{n}_{\eta^0_{\pi,1}} - \omega k_2 = 0 \text{ in } \Sigma^s_\infty.\\
& k_1 = 0 \text{ and } k_{1,x} = 0 \text{ on } \{0,L\}\times(0,+\infty),\\
& -\mathbf{w}^1_{c,t} - 2\Delta_z\mathbf{w}^1_c = -\sigma(\mathbf{w},q)\mathbf{n} \text{ on } \Sigma^i_\infty,\\
& -\mathbf{w}^2_{c,t} - \Delta_z\mathbf{w}^2_c = -\sigma(\mathbf{w},q)\mathbf{n} \text{ on } \Sigma^i_\infty,
\end{aligned} \tag{4.7.11}$$

with $\mathbf{w} = \Pi\mathbf{w} + \nabla N_s(k_2)$ and $q = -N_s(k_2) + N_{v^*}(\mathbf{w})$. Suppose that

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \mathcal{B}_e^* \begin{pmatrix} \Pi\mathbf{w} \\ k_1 \\ k_2 \\ \mathbf{w}_c^1 \\ \mathbf{w}_c^2 \end{pmatrix} = \begin{pmatrix} \mathbf{w}_c^1 - N_s((I+\overline{N}_s)^{-1}k_2)_{|\Gamma_i}\mathbf{n} \\ \mathbf{w}_c^2 - N_s((I+\overline{N}_s)^{-1}k_2)_{|\Gamma_i}\mathbf{n} \end{pmatrix},$$

where the second equality comes from Proposition 4.7.3). Then $\mathbf{w}_c^1 = \mathbf{w}_c^2 = N_s((I+\overline{N}_s)^{-1}k_2)_{|\Gamma_i}\mathbf{n}$. Setting $\mathbf{\Phi} = N_s((I+\overline{N}_s)^{-1}k_2)_{|\Gamma_i}\mathbf{n}$, the difference between the last two equations in (4.7.11) shows that

$$-\Delta_z \mathbf{w}_c^1 = 0 \text{ on } \Sigma_\infty^i.$$

Moreover, for all $t \geq 0$, $\mathbf{w}_c^1(t) \in \mathbf{H}^2(\Gamma_i) \cap \mathbf{H}_0^1(\Gamma_i)$. We deduce that $\mathbf{w}_c^1 = 0$ on Σ_∞^i. The last equation in (4.7.11) then shows that $\sigma(\mathbf{w}, q) = 0$ on Σ_∞^i. Finally, the conditions $\mathbf{w} = 0$ and $\sigma(\mathbf{w}, q)\mathbf{n} = 0$ on Σ_∞^i imply, using the unique continuation result proved in [22, 21], that $(\mathbf{w}, q) = (\mathbf{0}, 0)$ on $Q_\infty^{\pi,0}$. The Dirichlet boundary condition $\mathbf{w} = k_2\mathbf{e}_2$ on $\Sigma_\infty^{\pi,0}$ then ensures that $k_2 = 0$ on Σ_∞^s and the second equation of the beam provides provides $k_1 = 0$ on Σ_∞^s. We use Theorem 4.3.4 to conclude the proof. $\square$

Using Theorem 4.7.1, we can now state a stabilizability result on the system (4.7.1).

Theorem 4.7.2. For all $(\mathbf{v}^0, \hat{\eta}_1^0, \hat{\eta}_2^0) \in \mathbf{H}_{cc}$ there exits

$$\mathbf{v}_c \in \mathcal{C}^0([0,+\infty); \mathbf{H}^2(\Gamma_i) \cap \mathbf{H}_0^1(\Gamma_i)) \cap \mathcal{C}^1([0,+\infty); \mathbf{L}^2(\Gamma_i)),$$

satisfying

$$\sup_{t>0} \|\mathbf{v}_c(t)\|_{\mathbf{H}^2(\Gamma_i)\cap\mathbf{H}_0^1(\Gamma_i)} + \sup_{t>0} \|\mathbf{v}_{c,t}(t)\|_{\mathbf{L}^2(\Gamma_i)} < +\infty,$$

such that the classical solution $(\mathbf{v}, q, \hat{\eta}_1, \hat{\eta}_2)$ to (4.7.1), which belongs to

- $\mathbf{v} \in \mathcal{C}^0([0,+\infty); \mathbf{L}^2(\Omega_{\pi,0})) \cap \mathcal{C}^0((0,+\infty); \mathbf{H}^2(\Omega_{\pi,0})) \cap \mathcal{C}^1((0,+\infty); \mathbf{L}^2(\Omega_{\pi,0}))$,
- $q \in \mathcal{C}^0((0,+\infty); H^1(\Omega_{\pi,0}))$,
- $\hat{\eta}_1 \in \mathcal{C}^0([0,+\infty); H_0^2(\Gamma_s)) \cap \mathcal{C}^0((0,+\infty); H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \cap \mathcal{C}^1((0,+\infty); H_0^2(\Gamma_s))$,
- $\hat{\eta}_2 \in \mathcal{C}^0([0,+\infty); L^2(\Gamma_s)) \cap \mathcal{C}^0((0,+\infty); H_0^2(\Gamma_s)) \cap \mathcal{C}^1((0,+\infty); L^2(\Gamma_s))$,

satisfies, for all $a > 0$,

$$\sup_{t>a} \|(\mathbf{v}(t), \hat{\eta}_1(t), \hat{\eta}_2(t))\|_{\mathbf{H}^2(\Omega_{\pi,0})\times H^4(\Gamma_s)\times H^2(\Gamma_s)} < +\infty.$$

Remark 4.7.1. The unique continuation result of [22, 21] is a local argument. Hence, in the previous analysis, the control $\mathbf{v}_c$ can be localised on an open set $\Gamma_c \subset \Gamma_i$.

4.8 Appendix

We gather in this appendix a few general results used in the rest of the chapter.

Lemma 4.8.1. Let F_1, F_2 be Banach spaces, $\rho \in (0,1)$ and $\Phi \in \mathcal{C}^\rho([0,T]; \mathrm{Iso}(F_1, F_2))$. Then the map Φ^{-1} defined as $\Phi^{-1} : [0,T] \mapsto \Phi(t)^{-1}$ belongs to $\mathcal{C}^\rho([0,T]; \mathrm{Iso}(F_2, F_1))$.

Proof. $\mathrm{Iso}(F_1, F_2)$ is an open set of the Banach space $\mathcal{L}(F_1, F_2)$ and the map

$$\mathrm{Inv} : \begin{cases} \mathrm{Iso}(F_1, F_2) \to \mathrm{Iso}(F_2, F_1) \\ \qquad\qquad A \mapsto A^{-1} \end{cases}$$

is $\mathcal{C}^\infty$. Since $\Phi([0,T])$ is compact in $\mathrm{Iso}(F_1, F_2)$, we easily deduce that Inv is Lipschitz continuous on $\Phi([0,T])$, with constant C_{-1}. This shows that, for $t \neq s$,

$$\frac{\|\mathrm{Inv}(\Phi(t)) - \mathrm{Inv}(\Phi(s))\|_{\mathcal{L}(F_1,F_2)}}{|t-s|^\rho} \leq C_{-1} \frac{\|\Phi(t) - \Phi(s)\|_{\mathcal{L}(F_2,F_1)}}{|t-s|^\rho}$$

and the proof is complete since $\mathrm{Inv}(\Phi(r)) = \Phi^{-1}(r)$ by definition. $\square$

Lemma 4.8.2. Let (E_1, E_2) and (F_1, F_2) be two pairs of densely injected Banach spaces. Let $\Phi \in \mathrm{Iso}(E_1, F_1) \cap \mathrm{Iso}(E_2, F_2)$. Then, for all $\theta \in (0,1)$, $\Phi \in \mathrm{Iso}([E_1, E_2]_\theta, [F_1, F_2]_\theta)$.

Proof. By property of the interpolation we already know that $\Phi \in \mathcal{L}([E_1, E_2]_\theta, [F_1, F_2]_\theta)$. The only thing to prove is that the isomorphism property is preserved. Consider, for $y \in [F_1, F_2]_\theta$, the equation $\Phi x = y$. Since $[F_1, F_2]_\theta \subset F_2$ and $\Phi \in \mathrm{Iso}(E_2, F_2)$ there exists a unique $x \in E_2$ such that $\Phi x = y$. Moreover $\Phi^{-1} \in \mathrm{Iso}(F_1, E_1) \cap \mathrm{Iso}(F_2, E_2)$ and thus, by interpolation, $\Phi^{-1} \in \mathcal{L}([F_1, F_2]_\theta, [E_1, E_2]_\theta)$. Thus $x = \Phi^{-1} y \in [E_1, E_2]_\theta$, which concludes the proof. $\square$

For $\theta \geq 0$ consider the space $\mathbf{H}^\theta_{\mathrm{tr}}(\Omega_{\pi,0}) := \{\mathbf{u} \in \mathbf{H}^\theta(\Omega_{\pi,0}) \mid \mathbf{u} = 0 \text{ on } \Gamma_d,\ u_2 = 0 \text{ on } \Gamma_o\}$. Reasoning as in the proof of [38, Theorem 11.1] we see that

$$\mathbf{H}^\theta_{\mathrm{tr}}(\Omega_{\pi,0}) = \mathbf{H}^\theta(\Omega_{\pi,0}) \text{ if } 0 \leq \theta < \tfrac{1}{2}. \tag{4.8.1}$$

Lemma 4.8.3. For $\theta \in (0,1)$, recalling the definition (4.2.3) of V, the following equality holds

$$[V, \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta} = \mathbf{H}^\theta_{\mathrm{tr}}(\Omega_{\pi,0}) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}). \tag{4.8.2}$$

In particular, if $\theta \in [0, \frac{1}{2})$,

$$[V, \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta} = \mathbf{V}^\theta_{n,\Gamma_d}(\Omega_{\pi,0}). \tag{4.8.3}$$

Proof. We first remark that V is equal to $\mathbf{H}^1_{\mathrm{tr}}(\Omega_{\pi,0}) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$. As $\mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}) = \Pi\mathbf{L}^2(\Omega_{\pi,0})$ we can use [60, Theorem 1, Section 1.17.1] to obtain

$$\begin{aligned}
&[\mathbf{H}^1_{\mathrm{tr}}(\Omega_{\pi,0}) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}); \mathbf{L}^2(\Omega_{\pi,0}) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})]_{1-\theta} \\
&\qquad = [\mathbf{H}^1_{\mathrm{tr}}(\Omega_{\pi,0}), \mathbf{L}^2(\Omega_{\pi,0})]_{1-\theta} \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}) = \mathbf{H}^\theta_{\mathrm{tr}}(\Omega_{\pi,0}) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0}).
\end{aligned}$$

This concludes the proof of (4.8.2). The relation (4.8.3) simply follows from (4.8.2), (4.8.1) and the definition of $\mathbf{V}^\theta_{n,\Gamma_d}(\Omega_{\pi,0}) = \mathbf{H}^\theta(\Omega_{\pi,0}) \cap \mathbf{V}^0_{n,\Gamma_d}(\Omega_{\pi,0})$. □

The next lemma investigates the Hölder regularity in time of the operator L_1. We also use a transposition method to extend the result for weaker data.

Lemma 4.8.4. The operator L_1 defined by (4.4.16) belongs to

$$\mathcal{C}^\rho_\sharp([0,T]; \mathcal{L}(H^1_{\mathrm{lift}}(\Omega_{\pi,0}) \times \mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0}), \mathbf{H}^2(\Omega_{\pi,0}))),$$

and can be extended as an operator in

$$\mathcal{C}^\rho_\sharp([0,T]; \mathcal{L}((H^1(\Omega_{\pi,0})' \times (\mathcal{H}^{1/2}(\Gamma_{\pi,0}))'; \mathbf{L}^2(\Omega_{\pi,0}))).$$

Proof. The lifting operator L depends on time and we specify this dependency with the notation $L(t)(\cdot,\cdot)$. For $(w,\mathbf{g}) \in H^1_{\mathrm{lift}}(\Omega_{\pi,0}) \times \mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0})$ and $0 \le s < t \le T$ the pair $(\mathbf{v}, q) = L(t)(w,\mathbf{g}) - L(s)(w,\mathbf{g})$ is solution to

$$\begin{aligned}
&\lambda_0\mathbf{v} - \nu\Delta\mathbf{v} + D_\pi(s)\mathbf{v} + \nabla q = (D_\pi(s) - D_\pi(t))L_1(t)(w,\mathbf{g}) \text{ in } \Omega_{\pi,0}, \\
&\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_{\pi,0}, \ \ \mathbf{v} = 0 \text{ on } \Gamma_{\pi,0}, \ \ \mathbf{v} = 0 \text{ on } \Gamma_i, \\
&v_2 = 0 \text{ and } q = 0 \text{ on } \Gamma_o, \ \ \mathbf{v} = 0 \text{ on } \Gamma_b.
\end{aligned}$$

The uniform-in-time estimates on the periodic solution $(\overline{\mathbf{u}}_\pi, p_\pi, \eta_\pi)$ show that the constant C in Theorem 4.4.1 does not depends on time. Hence, the pair $(\mathbf{v}, q)$ satisfies the following estimate

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega_{\pi,0})} + \|q\|_{H^1(\Omega_{\pi,0})} \le C \left\|(D_\pi(s) - D_\pi(t))L_1(t)(w,\mathbf{g})\right\|_{\mathbf{L}^2(\Omega_{\pi,0})}.$$

The Hölder regularity of the periodic solution implies $D_\pi(\cdot) \in \mathcal{C}^\rho_\sharp([0,T]; \mathcal{L}(V, \mathbf{L}^2(\Omega_{\pi,0})))$, which yields

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{H}^2(\Omega_{\pi,0})} + \|q\|_{H^1(\Omega_{\pi,0})} &\le C|t-s|^\rho \left\|L_1(t)(w,\mathbf{g})\right\|_{\mathbf{H}^2(\Omega_{\pi,0})} \\
&\le C^2|t-s|^\rho(\|w\|_{H^1(\Omega_{\pi,0})} + \|\mathbf{g}\|_{\mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0})}),
\end{aligned}$$

where we used Theorem 4.4.1 again to estimate $\|L_1(t)(w,\mathbf{g})\|_{\mathbf{H}^2(\Omega_{\pi,0})}$. The first part of the lemma is proved.

To extend the operator L_1 we use a transposition method. For $t \geq 0$ and $\Phi \in \mathbf{L}^2(\Omega_{\pi,0})$ let $(\boldsymbol{\varphi}(t), \gamma(t)) \in \mathbf{H}^2(\Omega_{\pi,0}) \times H^1(\Omega_{\pi,0})$ be the solution to

$$\begin{aligned}
&\lambda_0 \boldsymbol{\varphi}(t) - \nu \Delta \boldsymbol{\varphi}(t) + D_\pi^a(t) \boldsymbol{\varphi}(t) + \nabla \gamma(t) = \Phi \ \text{ in } \Omega_{\pi,0}, \\
&\operatorname{div} \boldsymbol{\varphi}(t) = 0 \ \text{ in } \Omega_{\pi,0}, \ \ \boldsymbol{\varphi}(t) = 0 \ \text{ on } \Gamma_{\pi,0}, \ \ \boldsymbol{\varphi}(t) = 0 \ \text{ on } \Gamma_i, \\
&\varphi_2(t) = 0 \ \text{ and } \gamma(t) = (\overline{\mathbf{u}}_\pi(t) \cdot \mathbf{n}) \varphi_1(t), \ \text{ on } \Gamma_o, \ \ \boldsymbol{\varphi}(t) = 0 \ \text{ on } \Gamma_b.
\end{aligned} \tag{4.8.4}$$

For $(w, \mathbf{g}) \in H^1_{\text{lift}}(\Omega_{\pi,0}) \times \mathcal{H}^{3/2}_{00}(\Gamma_{\pi,0})$ set $(\mathbf{u}(t), p(t)) = L(t)(w, \mathbf{g})$. Using the Green formula (see the proof of Lemma 4.5.4), we obtain

$$\begin{aligned}
\int_{\Omega_{\pi,0}} \mathbf{u}(t) \cdot \Phi &= \int_{\Omega_{\pi,0}} \mathbf{u}(t) \cdot (\lambda_0 \boldsymbol{\varphi}(t) - \nu \Delta \boldsymbol{\varphi}(t) + D_\pi^a(t) \boldsymbol{\varphi}(t) + \nabla \gamma(t)) \\
&= \int_{\Gamma_{\pi,0}} \mathbf{g} \cdot \sigma(\boldsymbol{\varphi}(t), \gamma(t))\, \mathbf{n} - \int_{\Omega_{\pi,0}} w\, \gamma(t).
\end{aligned}$$

For $(w, \mathbf{g}) \in (H^1(\Omega_{\pi,0}))' \times (\mathcal{H}^{1/2}(\Gamma_{\pi,0}))'$ consider the following variational formulation:

$$\begin{aligned}
&\text{Find } \mathbf{u}(t) \in \mathbf{L}^2(\Omega_{\pi,0}) \text{ such that} \\
&\int_{\Omega_{\pi,0}} \mathbf{u}(t) \cdot \Phi = \langle \mathbf{g}, ((\sigma(\boldsymbol{\varphi}(t), \gamma(t))\, \mathbf{n}) \cdot \mathbf{e}_2 \rangle_{(\mathcal{H}^{1/2}(\Gamma_{\pi,0}))', \mathcal{H}^{1/2}(\Gamma_{\pi,0})} \\
&\qquad\qquad - \langle w, \gamma(t) \rangle_{(H^1(\Omega_{\pi,0}))', H^1(\Omega_{\pi,0})},
\end{aligned} \tag{4.8.5}$$

for all $\Phi \in \mathbf{L}^2(\Omega_{\pi,0})$ and with $(\boldsymbol{\varphi}(t), \gamma(t))$ solution to (4.8.4). Using a density argument as in [55, Theorem A.1] we can prove that (4.8.5) admits a unique solution. Hence we have extended the operator L_1 on $(H^1(\Omega_{\pi,0}))' \times (\mathcal{H}^{1/2}(\Gamma_{\pi,0}))'$. To prove the Hölder regularity in time of L_1 on the weaker space we note that, for $0 \leq s < t$, $\mathbf{v} = L_1(t)(w, \mathbf{g}) - L_1(s)(w, \mathbf{g})$ satisfies

$$\begin{aligned}
\int_{\Omega_{\pi,0}} \mathbf{v} \cdot \Phi = \langle \mathbf{g}, ((\sigma(\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(s), \gamma(t) - \gamma(s))\, \mathbf{n}) \cdot \mathbf{e}_2 \rangle_{(\mathcal{H}^{1/2}(\Gamma_{\pi,0}))', \mathcal{H}^{1/2}(\Gamma_{\pi,0})} \\
- \langle w, \gamma(t) - \gamma(s) \rangle_{(H^1(\Omega_{\pi,0})', H^1(\Omega_{\pi,0})},
\end{aligned}$$

for all $\Phi \in \mathbf{L}^2(\Omega_{\pi,0})$ and $(\boldsymbol{\varphi}(t), \gamma(t))$ (respectively $(\boldsymbol{\varphi}(s), \gamma(s))$) solution to (4.8.4) at time t (respectively at time s). We then write

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{L}^2(\Omega_{\pi,0})} &= \sup_{\Phi \in \mathbf{L}^2(\Omega_{\pi,0})} \left| \int_{\Omega_{\pi,0}} \mathbf{v} \cdot \Phi \right| \\
&\leq C \|\mathbf{g}\|_{(\mathcal{H}^{1/2}(\Gamma_{\pi,0}))'} \Big(\|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(s)\|_{\mathbf{H}^2(\Omega_{\pi,0})} + \|\gamma(t) - \gamma(s)\|_{H^1(\Omega_{\pi,0})} \Big) \\
&\quad + C \|w\|_{(H^1(\Omega_{\pi,0}))'} \|\gamma(t) - \gamma(s)\|_{H^1(\Omega_{\pi,0})}.
\end{aligned} \tag{4.8.6}$$

The adjoint system (4.8.4) satisfies the same estimates as (4.4.1) and the Hölder dependency of the solution is similar. Hence,

$$\|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(s)\|_{\mathbf{H}^2(\Omega_{\pi,0})} + \|\gamma(t) - \gamma(s)\|_{H^1(\Omega_{\pi,0})} \leq C|t - s|^\rho \|\Phi\|_{\mathbf{L}^2(\Omega_{\pi,0})},$$

which, coupled with (4.8.6), concludes the proof. □

Chapter 5

Conclusion and perspectives

In this thesis, we studied fluid–structure models with boundary conditions involving the pressure. To conclude this thesis, we present a brief summary of each chapter and give some perspectives related to this work.

Fluid–structure system with boundary conditions involving the pressure

In the first chapter we developed the general framework required to study the Navier–Stokes equations with mixed Dirichlet/pressure boundary conditions. We also studied the fluid–structure model in the initial domain of the fluid. These techniques were used to prove the existence of local strong solutions without assumptions on the initial data, and of strong solutions on arbitrary time intervals for small data.

A natural extension of this work is to investigate the existence of global strong solutions without smallness assumptions on the initial data. This was done in [28] for a similar system with periodic conditions on the inflow and outflow boundaries. The proof, based on a blow-up alternative, requires additional estimates showing that the beam does not 'touch' the bottom of the fluid domain in finite time. It could be interesting to adapt these estimates for pressure boundary conditions.

Existence of time-periodic solutions to a fluid–structure system

The second part of this thesis was dedicated to the study of periodic behaviours for fluid–structure system. When the system is subjected to small T-periodic source terms, where $T > 0$ is arbitrary, we prove the existence of T-periodic responses. The solution that we obtain has different time regularity properties depending on the regularity of the sources. In particular we prove the existence of solutions with Hölder regularity in time. Obtaining a solution with this regularity was directly motivated by the stabilization question for this system. However, as seen in Chapter 4, the regularity that we obtain is not sufficient for the stabilization analysis of the fluid–structure system around this periodic solution (unless we assume that this periodic solution is small enough to be stabilized by the feedback law coming from the linearization around the zero solution, a situation which is much easier to study).

This limitation is only due to the non-homogeneous divergence when we stabilize the system in a neighbourhood of a periodic solution. Two options are then possible: try to remove this divergence condition using, for example, a different change of variables, or increase the time-regularity of the periodic solution constructed in this section. The second option seems complicated as the time-regularity is related to the continuity condition in the fluid–structure system, which does not provide the continuity of higher derivatives.

Another open question concerns the uniqueness of the periodic solution. The strategy used in Chapter 2 to prove the uniqueness was to start with the local uniqueness provided by the Banach fixed point and to use a 'Cauchy–Lipschitz' continuation technique. For the periodic solution, the fixed point argument provides the uniqueness only for sufficiently small solutions. It is not clear if other periodic solutions, with higher energy, could exist. One idea may come from the energy estimate for the periodic system. Consider the fluid–structure system studied in Chapter 3 with inflow and outflow boundary conditions involving the pressure (and a source term $\omega_{i/o}$ on $\Gamma_{i/o}$). The following energy identity is proved in [49]

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega_{\eta(t)})}+\|\partial_t\eta\|^2_{L^2(\Gamma_s)}+\|\partial_x\eta\|^2_{L^2(\Gamma_s)}+\|\partial_{xx}\eta\|^2_{L^2(\Gamma_s)}\right]\\&+\frac{\mu}{2}\left\|\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right\|^2_{\mathbf{L}^2(\Omega_{\eta(t)})}+\|\partial_{tx}\eta\|^2_{L^2(\Gamma_s)}=\int_{\Gamma_{i/o}}\omega_{i/o}\,u_1.\end{aligned}$$

Integrating the previous identity over a period shows that the dissipative part of the system exactly compensate the energy coming from the source terms (as expected for a periodic system). This information may be use to recover the uniqueness of a periodic solution.

Stabilization of a time-periodic fluid–structure system

The last chapter of the thesis is a first and important step towards the stabilization of the fluid–structure system in a neighbourhood of a non-small periodic solution. The stabilization of the nonlinear system is ongoing. Our next objective is to obtain the control under a feedback form. The usual method consists in considering the associated quadratic problem and to solve an infinite dimensional Riccati equation. For time-dependent operators with constant domains this was investigated in [19]. In our case, the operator $\mathcal{A}(t)$ does not satisfy the assumptions considered in [19] and the proofs need to be adapted. Another strategy is to used the Floquet representation to autonomize the unstable part of the system. Using the notation of Section 4.3.3 in Chapter 4, we consider the operator $B := \frac{-1}{T}\log(V_{|X_u(0)}(0))$ (B is well defined as the operator V has discrete spectrum and we can define a branch of the logarithm defined on a neighbourhood of $\{\lambda_j\}_{0\leq j\leq N}$). We then introduce the operator $Q(t) = U_u(t,0)e^{tB}$ for $t \in \mathbb{R}$. The Floquet representation of the unstable part is given by the relation

$$U_u(t,s) = Q(t)e^{(t-s)B}Q(s)^{-1}.$$

The operator U_u is then associated to the evolution operator of the time-independent equation $y' + By = 0$ on $X_u(0)$. This reduction enables the study of the stabilization by feedback law on a time-independent, finite-dimensional system. This approach is developped in [8] to recover a finite-dimensional feedback law.

Another component of the stabilization analysis of the nonlinear system is the choice of spaces in which to write a proper fixed point theorem. The nonlinear system is quasi-linear and requires maximal regularity results. Hence, we want to study the nonlinear system in the space of Hölder-in-time decaying functions. This Hölder regularity, up to $t = 0$, imposes compatibility conditions between the initial data and the nonlinear terms, which is almost impossible to satisfy as mentioned in [41]. The idea is to consider spaces of functions which are not necessarily continuous up to $t = 0$, but which satisfy the expected regularity on a time interval $[a, +\infty)$. These ideas were used in [41] to study the stability of fully nonlinear periodic systems and could be adapted to our case.

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Analyse et contrôle de systèmes fluide–structure avec conditions limites sur la pression

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Résumé : Le sujet de la thèse porte sur l'étude (existence, unicité, régularité) et le contrôle de problèmes fluide-structure possédant des conditions limites sur la pression. Le système étudié couple une partie fluide, décrite par les équations de Navier–Stokes incompressibles dans un domaine 2D et une partie structure, décrite par une équation 1D de poutre amortie située sur une partie du bord du domaine fluide. Dans le Chapitre 2, on étudie l'existence de solutions fortes pour ce modèle. Nous démontrons des résultats de régularité optimale pour le système de Stokes avec conditions de bord mixtes sur un domaine non régulier. Ces résultats sont ensuite utilisés pour prouver l'existence et l'unicité de solutions fortes, locales en temps, pour le système fluide-structure sans hypothèse de petitesse sur les données initiales. Le Chapitre 3 réutilise l'analyse précédente dans le cadre de solutions périodiques en temps. Nous développons un critère d'existence de solutions périodiques pour un problème parabolique abstrait. Ce critère est ensuite appliqué au système fluide-structure et nous obtenons l'existence de solutions strictes, périodiques et régulières en temps, pour des termes sources périodiques suffisamment petits. Le quatrième volet de la thèse porte sur la stabilisation du système fluide-structure au voisinage d'une solution périodique. Le système linéarisé sous-jacent est décrit à l'aide d'un opérateur $\mathrm{A(t)}$ dont le domaine dépend du temps. Nous démontrons l'existence d'un opérateur parabolique d'évolution pour ce système linéaire. Cet opérateur est ensuite utilisé, dans le cadre de la théorie de Floquet, pour étudier le comportement asymptotique du système. Nous adaptons la théorie existante pour des opérateurs à domaine constant au cas de domaine non constant. Nous obtenons la stabilisation exponentielle du système linéaire à l'aide d'un contrôle sur la frontière du domaine fluide.

Mots-clés : Intéraction fluide–structure, contrôle frontière, stabilisation, équations de Navier–Stokes, équation de poutre, conditions de bord sur la pression, systèmes périodiques en temps.

Abstract : In this thesis we study the well-posedness (existence, uniqueness, regularity) and the control of fluid-structure system with boundary conditions involving the pressure. The fluid part of the system is described by the incompressible Navier-Stokes equations in a 2D rectangular type domain coupled with a 1D damped beam equation localised on a boundary part of the fluid domain. In Chapter 2 we investigate the existence of strong solutions for this model. We prove optimal regularity results for the Stokes system with mixed boundary conditions in non-regular domains. These results are then used to obtain the local-in-time existence and uniqueness of strong solutions for the fluid-structure system without smallness assumption on the initial data. Chapter 3 uses the previous analysis in the framework of periodic (in time) solutions. We develop a criteria for the existence of periodic solutions for an abstract parabolic system. This criteria is then used on the fluid-structure system to prove the existence of a periodic and regular in time strict solution, provided that the periodic source terms are small enough. In Chapter 4 we study the stabilisation of the fluid-structure system in a neighbourhood of a periodic solution. The underlying linear system involves an operator $\mathrm{A(t)}$ with a domain which depends on time. We prove the existence of a parabolic evolution operator for this linear system. This operator is then used to apply the Floquet theory and to describe the asymptotic behaviour of the system. We adapt the known results for an operator with constant domain to the case of operators with non constant domain. We obtain the exponential stabilisation of the linear system with control acting on a part of the boundary of the fluid domain.

Keywords : Fluid–structure interaction, boundary control, stabilization, Navier–Stokes equations, beam equation, pressure boundary conditions, time-periodic systems.