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Constructing Free Resolutions of Cohomology Algebras

Ahsan Ahmed Jaleel, BSc, MSc.

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School of Applied Sciences and Engineering
Faculty of Science

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Abstract

We define the $\mathcal{H}(\mathcal{R})$-algebra of a space as the algebraic object consisting of the graded cohomology groups of the space with coefficients in a general ring $\mathcal{R}$, together with all primary cohomology operations on these groups, subject to the relations between the operations. This structure can be encoded as a functor from the category $\mathcal{H}(\mathcal{R})$ containing products of Eilenberg-Mac Lane spaces over $\mathcal{R}$ to the category of pointed sets.

The free $\mathcal{H}(\mathcal{R})$-algebras are the $\mathcal{H}(\mathcal{R})$-algebras of a product of Eilenberg-Mac Lane spaces. In this thesis we show how to construct free simplicial resolutions of $\mathcal{H}(\mathcal{R})$-algebras using the free and underlying functors.

Given a space X, we also construct a cosimplicial space such that the cohomology of this cosimplicial space is a free simplicial resolution of the $\mathcal{H}(\mathcal{R})$-algebra of X. For $\mathcal{R} = \mathbb{F}_p$, the finite field on p elements, this cosimplicial resolution fits the E^2 page of a spectral sequence and give convergence results under certain finiteness restrictions on X. For $\mathcal{R} = \mathbb{Z}$, the integers, a similar result is not obtained and the reasons for this are given.

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Contents

Chapter 1

Introduction

In this thesis we construct resolutions of the algebraic object consisting of the cohomology groups of a $\mathcal{CW}_*-$complex with coefficients in an arbitrary ring with unity $\mathcal{R}$, together with all the primary cohomology operations acting on this graded group. The algebras over the Steenrod algebra is an example of this kind of algebraic object for $\mathcal{R} = \mathbb{F}_2$.

We take an approach that is Eckmann-Hilton dual to that of Stover's work on Π-algebras [53]. The Stover construction has led to a number of applications in homotopy theory. These inlcude the development of resolution model categories by Dwyer, Kan and Stover [22], some new spectral sequences [53, 36] and homotopy calculations [27].

The Eckmann-Hilton dual to Π-algebras appeared in [46] but that was for $\mathcal{R} = \mathbb{Z}$ only and was called an $\mathcal{H}$-algebra. To allow for arbitrary rings we will call them $\mathcal{H}(\mathcal{R})$-algebras, so that an $\mathcal{H}(\mathbb{F}_2)$-algebra would be an algebra over the Steenrod algebra. The structure of $\mathcal{H}(\mathcal{R})$-algebras can be encoded in a functor from $\mathcal{H}(\mathcal{R})$, the category of products of Eilenberg-Mac Lane spaces over $\mathcal{R}$ to the category of pointed sets. In fact, in the terminology of Borceux [13], Lawvere [39] and Ehresmann [25, 24] $\mathcal{H}(\mathcal{R})$-algebras can be thought of as models of a product sketch.

The category of $\mathcal{H}(\mathcal{R})$-algebras do not form an abelian category so to do homotopy theory on them we need to define a model category structure and work with free simplicial resolutions. We define model category structures in Chapter 4 and using the model category structure of Section 4.1.2, we are able to prove the existence of free simplicial resolutions of $\mathcal{H}(\mathcal{R})$-algebras in Chapter 5.

In Chapter 5, the simplicial construction formed by using the free and underlying functors together with the natural transformations (counit and unit of adjunction) is proven to be a resolution. Our proof holds for any model category with free and underlying functors. This is analogous to the construction of resolutions by Huber's standard method [35] with slight modifications. We show in Chapter 3 that for $\mathcal{R} = \mathbb{Z}$ we also need an infinite product sketch and give results for a model category on these models in Section 4.1.2.

The other model category structure (Section 4.3) is Bousfield's resolution model category on cosimplicial spaces [15] allowing a comparison of the free cosimplicial space, constructed in Chapter 6, with the Bousfield-Kan resolution on simplicial sets ([16], I 4.1). The resolution of Chapter 6 using Eilenberg-Mac Lane spaces is generally infinite dimensional, even if the space is finite and will generally be connected, even if the space was not. For $\mathcal{R} = \mathbb{Z}$ it is not possible to show a $\mathcal{G}$-equivalence with Bousfield-Kan resolution because the resolution by products of Eilenberg-Mac Lane spaces is not acyclic for all abelian group coefficients.

Nevertheless for $\mathcal{R} = \mathbb{F}_p$ and for a space X with finitely generated cohomology groups the $\mathcal{G}$-equivalence with Bousfield-Kan resolution allows some applications. Using the vector dual and working with homology with homology co-operations, the resolution by Eilenberg-Mac Lane spaces fits into the E^2 page of the Bousfield homology spectral sequence converging to the homology of X which then may be redualized to give information on cohomology.

The spectral sequence can also be applied to the mapping space $\mathrm{map}_*(Y, X)$ for a fixed finite space Y. Setting Y to be a circle gives a spectral sequence converging to the homology of the loops on X.

Chapter 2

Preliminaries

2.1 Basic category theory and definitions

2.1.1 Natural transformations

In this thesis for composition of morphisms in a general category $\mathcal{C}$, fg is used to denote g following f.

Definition 2.1.1. *Let $L, R : \mathcal{C} \to \mathcal{D}$ be covariant functors. A* ***natural transformation*** *([41], I.4) ϑ from L to R is a class of morphisms, $\vartheta_X : LX \to RX$, such that for each object $X \in \mathcal{C}$ and for all morphisms $f : X \to Y$ in $\mathcal{C}$, the following diagram is commutative.*

$$\begin{tikzcd} LX \arrow[r, "\vartheta_X"] \arrow[d, "Lf"'] & RX \arrow[d, "Rf"] \\ LY \arrow[r, "\vartheta_Y"'] & RY \end{tikzcd}$$

The composition of two natural transformations are defined in the obvious way, but we will make a short note of how natural transformations are defined on a composition of functors ([35], $\oint 1$).

2.1.2 Composition of natural transformations and functors

Let $\vartheta : L \longrightarrow R$ be a natural transformation, U and V be covariant functors such that the compostion of functors ULV and URV are defined. Then,

$$\begin{aligned} U\vartheta : UL \longrightarrow UR, &\quad \text{is defined as} \\ (U\vartheta)_Y := U(\vartheta_Y). \end{aligned} \tag{2.1}$$

$$\begin{aligned} \vartheta V : LV \longrightarrow RV, &\quad \text{is defined as} \\ (\vartheta V)_X := \vartheta_{VX}. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) we define

$$\begin{aligned} U\vartheta V : ULV \longrightarrow URV, &\ \text{ as} \\ (U\vartheta V)_X := U\vartheta_{VX}. \end{aligned} \tag{2.3}$$

Fact 2.1.2. ([35], $\oint 1$) Let U and V be functors and ϑ and ϑ' be natural transformations, then

$$(U\vartheta V)(U\vartheta' V) = U(\vartheta\vartheta')V \tag{2.4}$$

Lemma 2.1.3. *For any two natural transformations $\phi : L \longrightarrow R$ and $\psi : M \longrightarrow N$ the following identities hold when the respective compositions are defined ([28], Appendix) or ([35], $\oint 1$).*

$$\begin{tikzcd} ML \arrow[r, "M\phi"] \arrow[d, "\psi L"'] & MR \arrow[d, "\psi R"] \\ NL \arrow[r, "N\phi"'] & NR \end{tikzcd}$$

$$(\psi R)(M\phi) = (N\phi)(\psi L) \tag{2.5}$$

$$\begin{tikzcd}
LM \arrow[r, "\phi M"] \arrow[d, "L\psi"'] & RM \arrow[d, "R\psi"] \\
LN \arrow[r, "\phi N"'] & RN
\end{tikzcd}$$

$$(R\psi)(\phi M) = (\phi N)(L\psi) \tag{2.6}$$

Example 2.1.4. Let $L = FU^i$, $R = FU^{i+1}$, $M = FU$, $N = I$, $\phi = hFU^i$, $\psi = \epsilon$ in equation (2.6), then we have

$$\begin{tikzcd}
FU^iFU \arrow[r, "hFU^iFU"] \arrow[d, "FU^i\epsilon"'] & FU^{i+1}FU \arrow[d, "FU^{i+1}\epsilon"] \\
FU^iI \arrow[r, "hFU^iI"'] & FU^{i+1}I
\end{tikzcd}$$

Then

$$FUFU^i\epsilon \ hFU^iFU = hFU^i \ \ FU^i\epsilon \tag{2.7}$$

Example 2.1.5. Let $L = FU$, $R = FU^2$, $M = FU^i$, $N = FU^{i+1}$, $\phi = \nu$, $\psi = hFU^i$ in equation (2.5), then we have

$$\begin{tikzcd}
FU^iFU \arrow[r, "FU^i\nu"] \arrow[d, "hFU^iFU"'] & FU^iFU^2 \arrow[d, "hFU^iFU^2"] \\
FU^{i+1}FU \arrow[r, "FU^{i+1}\nu"'] & FU^{i+1}FU^2
\end{tikzcd}$$

Then

$$FUFU^i\nu \ hFU^iFU = hFU^iFU^2 \ \ FU^i\nu \tag{2.8}$$

2.1.3 Adjoints

In this section, we explain the concept of adjoint functors between categories that will be used to define free objects in a category. There are two common ways to define the adjunction of functors. Definition 2.1.7 is given using natural transformations called unit and counit of adjunction and Definition 2.1.8 is described using Hom sets ([41], IV.1).

Notation 2.1.6. *We will use id_X to denote the identity map from an object X to itself within a category. The identity functor will be denoted by I and the identity natural transformation on a functor G is denoted by 1_G.*

Definition 2.1.7. *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. L is left **adjoint** to R, if and only if there exists natural transformations,*

*(i) $\eta : I \to RL$, called the **unit of adjunction** and*

*(ii) $\epsilon : LR \to I$ called the **counit of adjunction** such that the following two diagrams commute.*

$$\begin{tikzcd} & L \\ L \arrow[ur, "1_L"] \arrow[r, "L\eta"'] & LRL \arrow[u, "\epsilon L"'] \end{tikzcd} \qquad \begin{tikzcd} RLR \arrow[r, "R\epsilon"] & R \\ R \arrow[u, "\eta R"] \arrow[ur, "1_R"'] & \end{tikzcd} \tag{2.9}$$

Definition 2.1.8. *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. L is left **adjoint** to R, if there exists a family of bijections*

$$Hom_{\mathcal{D}}(LX, Y) \cong Hom_{\mathcal{C}}(X, RY) \tag{2.10}$$

which is natural in X and Y.

Notation 2.1.9. *We will use $\mathcal{SET}$ to denote the category of sets, $\mathfrak{G}rp$ for the category groups and $\mathcal{A}b$ for the category of abelian groups.*

Definition 2.1.10. *Let $\mathcal{C}$ be a category whose objects are sets with some additional structure and morphisms of $\mathcal{C}$ respecting this structure. A functor which "forgets" some or all the structure of the objects in $\mathcal{C}$ is called an underlying functor ([41], I.3).*

Definition 2.1.11. *Let $\mathcal{C}$ be a category where the underlying functor U is defined on $\mathcal{C}$. Then F is called a* ***free functor*** *if F is left adjoint to U. We denote $F \dashv U$ to mean F is left adjoint to U.*

Example 2.1.12. Let U be the underlying functor from $\mathfrak{G}rp \to \mathcal{SET}$ which send a group $G \in \mathfrak{G}rp$ to the underlying set of G. The free functor F from $\mathcal{SET} \to \mathfrak{G}rp$ sends a set $A \in \mathcal{SET}$ to the free group generated on the elements of A.

2.1.4 Categorical Duality

To each category $\mathcal{C}$ there is an associated opposite category $\mathcal{C}^{op}$. The objects of $\mathcal{C}^{op}$ are the objects of $\mathcal{C}$. However, for any morphism $f : a \to b$ in $\mathcal{C}$, the corresponding morphism in $\mathcal{C}^{op}$ is defined as $f^{op} : b \to a$ (the direction of the arrow reversed). Also the composition $f^{op}g^{op} = (gf)^{op}$ in $\mathcal{C}^{op}$. According to this, a contravariant functor $S : \mathcal{C} \to \mathcal{D}$ can be regarded as a covariant functor $S : \mathcal{C}^{op} \to \mathcal{D}$. Therefore, for every statement in a category $\mathcal{C}$ that can be expressed as a diagram, there is a corresponding statement in the opposite category $\mathcal{C}^{op}$ by reversing arrows and order of composition just as explained above. This procedure of interchanging arrows and order of composition is called **categorical duality**.

2.1.5 Limits

Next, we define a limit in a category. Terminal objects, products, equalizers and pullbacks can be unified using the concept of limit.

Definition 2.1.13. *Let L be a functor from $\mathcal{D}$ to $\mathcal{C}$, a* ***cone*** *([12], 2.6.1) on L consists of an object C in $\mathcal{C}$ and for every object $D_1 \in \mathcal{D}$, a morphism $t_{D_1} : C \longrightarrow LD_1$ in $\mathcal{C}$, such that, for every morphism $d : D_1 \longrightarrow D_2$ in $\mathcal{D}$, $t_{D_2} = L(d)t_{D_1}$.*

Notation 2.1.14. *We will denote a cone on a functor L by $(C, (t_{D_i})_{D_i \in \mathcal{D}})$.*

Definition 2.1.15. *A* ***limit*** *([12], 2.6.2), if it exists, of a functor $R : \mathcal{D} \longrightarrow \mathcal{C}$ is a cone $(L, (t_D)_{D\in\mathcal{D}})$ on R such that, for every cone $(M, (q_D)_{D\in\mathcal{D}})$ on R, there exists a unique morphism $m : M \longrightarrow L$ such that for every $D \in \mathcal{D}$, with $q_D = t_D m$. The limit of a functor is also the terminal object in the category of cones on $\mathcal{C}$.*

Remark 2.1.16. *Let $\mathcal{D}$ be an ordered category, the limit of the functor $R : \mathcal{D} \longrightarrow \mathcal{C}$, if it exists will be denoted by $\lim_{d\in\mathcal{D}} X_d$, where $X_d \in \mathcal{C}$. Dually the colimit if it exists will be denoted by $\mathrm{colim}_{d\in\mathcal{D}} X_d$, where $X_d \in \mathcal{C}$.*

Here as an example, we will show that products are a particular type of limit.

Example 2.1.17. Let $\mathcal{D}$ be the discrete category with only two elements and no non-identity morphisms. Also let $R : \mathcal{D} \longrightarrow \mathcal{C}$, so a cone on R is an object $A \in \mathcal{C}$ with morphisms $RD_1 \xleftarrow{q_1} A \xrightarrow{q_2} RD_2$, where $D_1, D_2 \in \mathcal{D}$. If the terminal cone $RD_1 \xleftarrow{t_1} L \xrightarrow{t_2} RD_2$ on R exists then for any other cone (C, c_1, c_2), there is a unique morphism of cones $u : (C, c_1, c_2) \longrightarrow (L, t_1, t_2)$ with $c_j = t_j u$, $j = 1, 2$, by the universal property of the terminal object. Thus, the following diagram commutes.

$$\begin{tikzcd}
 & C \arrow[dl, "c_1"'] \arrow[d, "u"] \arrow[dr, "c_2"] & \\
RD_1 & L \arrow[l, "t_1"] \arrow[r, "t_2"'] & RD_2
\end{tikzcd}$$

The limit (L, t_1, t_2), if it exists, in this case is called the binary product ([12], 2.1.1) of RD_1 and RD_2 and we denote it by $(RD_1 \times RD_2, t_1, t_2)$.

Definition 2.1.18. *Let J be a set and $\{C_i | i \in J\}$ be a family of objects in a category $\mathcal{C}$. A* ***product*** *([12], 2.1.4) of that family is an object $\prod_{i\in J} C_i$ together with morphisms $\{pr_i : \prod_{i\in J} C_i \longrightarrow C_i\}$ such that for any family of morphisms*

$\{f_i : Y \longrightarrow C_i | i \in J\}$ *there is a unique morphism* $\{f_i\}_{i\in J} : Y \longrightarrow \prod\limits_{i\in J} C_i$ *with* $pr_i\{f_i\}_{i\in J} = f_i$ *for all* $i \in J$. *Thus we have the commutative diagram*

$$\begin{tikzcd} & C_i \\ Y \arrow[ur, "f_i"] \arrow[r, "\{f_i\}_{i\in J}"'] & \prod\limits_{i\in J} C_i \arrow[u, "pr_i"'] \end{tikzcd}$$

The categorical dual of a product is called a coproduct.

Definition 2.1.19. *Let* J *be a set and* $\{C_i | i \in J\}$ *be a family of objects in a category* $\mathcal{C}$. *A* ***coproduct*** *([12], 2.2.1) of that family is an object* $\coprod\limits_{i\in J} C_i$ *together with morphisms* $\{inc_i : C_i \longrightarrow \coprod\limits_{i\in J} C_i\}$ *such that for any family of morphisms* $\{f_i : C_i \longrightarrow Y | i \in J\}$ *there is a unique morphism* $< f_i >_{i\in J}: \coprod\limits_{i\in J} C_i \longrightarrow Y$ *with* $< f_i >_{i\in J}\ inc_i = f_i$ *for all* $i \in J$. *Thus we have the commutative diagram*

$$\begin{tikzcd} & C_i \arrow[dl, "f_i"'] \arrow[d, "inc_i"] \\ Y & \coprod\limits_{i\in J} C_i \arrow[l, "<f_i>_{i\in J}"] \end{tikzcd}$$

Definition 2.1.20. *In any category* $\mathcal{C}$ *and for a diagram of the form*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

the ***equalizer*** *([12], 2.4.1) of* f *and* g *is the object* E *and an arrow* $e : E \longrightarrow A$ *such that given any* $z : Z \longrightarrow A$ *with* $fz = gz$, *there is a unique* $u : Z \longrightarrow E$ *with* $eu = z$.

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dotted, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Definition 2.1.21. *If a category $\mathcal{C}$ has a zero object, $\mathfrak{z}$, then for any two objects $A, B \in \mathcal{C}$, the unique maps $A \longrightarrow \mathfrak{z}$ and $\mathfrak{z} \longrightarrow B$ have a composite $0 : A \longrightarrow \mathfrak{z} \longrightarrow B$, called the* ***zero map****.*

Definition 2.1.22. *In a general category $\mathcal{C}$, with zero object, the* ***kernel*** *of a map $f : A \longrightarrow B$ is a pair consisting of an object K and a map $k : K \longrightarrow A$ with $fk = 0$ (the zero map), such that, if there is any other map $z : Z \longrightarrow A$ with $fz = 0$ then there exists a unique $u : Z \longrightarrow K$ making the triangle $z = ku$ commute.*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[rr, bend left, "0"] & A \arrow[r, "f"] & B \\
Z \arrow[u, dotted, "u"] \arrow[ur, "z"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd} \tag{2.11}$$

The kernel in a general category is an instance of an equalizer. To see this, firstly in a category that has equalizers we let g be the zero map (factor g as $A \longrightarrow \mathfrak{z} \longrightarrow B$, given in Defintion 2.1.21) in the diagram of Definition 2.1.20. Then (K, k) is the equalizer of f and 0 and gives the kernel diagram (2.11).

Definition 2.1.23. *In a category $\mathcal{C}$ with maps f and g as follows $A \xrightarrow{f} C \xleftarrow{g} B$, the* ***pullback*** *([12], 2.5.1) of f and g consists of arrows $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ such that for any $z_1 : Z \longrightarrow A$ and $z_2 : Z \longrightarrow B$ with $fz_1 = gz_2$, then there is a unique map $u : Z \longrightarrow P$ making the following diagram commute.*

$$\begin{tikzcd}
Z \arrow[dr, dotted, "u"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"'] \\
& A \arrow[r, "f"'] & C
\end{tikzcd} \tag{2.12}$$

That is $z_1 = p_1 u$ and $z_2 = p_2 u$. Given a pullback square (2.12) the map p_1 is called a pullback of g along f ([32], 7.2.10).

Notation 2.1.24. *The pullback of diagram (2.12) is denoted by $A \times_C B$.*

Definition 2.1.25. *In a category $\mathcal{C}$ with maps f and g as follows $A \xleftarrow{f} C \xrightarrow{g} B$, the **pushout** ([12], 2.5.1) of f and g consists of arrows $A \xrightarrow{i_1} P \xleftarrow{i_2} B$ such that for any $z_1 : A \longrightarrow Z$ and $z_2 : B \longrightarrow Z$ with $z_1 f = z_2 g$, then there is a unique map $u : P \longrightarrow Z$ making the following diagram commute.*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "i_2"'] \arrow[ddr, bend left, "z_2"] & \\
A \arrow[r, "i_1"] \arrow[drr, bend right, "z_1"'] & P \arrow[dr, dotted, "u"] & \\
& & Z
\end{tikzcd} \tag{2.13}$$

That is $z_1 = ui_1$ and $z_2 = ui_2$. Given a pushout square (2.13) the map i_2 is called a pushout of f along g ([32], 7.2.10).

Notation 2.1.26. *The pushout of diagram (2.13) is denoted by $A \coprod_C B$.*

The following theorem further explains how products, equalizers and pullbacks are interrelated.

Theorem 2.1.27. *A category has finite products and equalizers if and only if it has pullbacks and a terminal object ([6], 5.16).*

Theorem 2.1.28. *A category has all limits of some cardinality iff it has all equalizers and products of that cardinality, where $\mathcal{C}$ has limits (resp. products) of cardinality κ if and only if $\mathcal{C}$ has a limit for every diagram $D : J_1 \longrightarrow \mathcal{C}$ where $card(J_1) \leq \kappa$ (resp. C has all products of κ many objects) ([6], 5.24).*

2.1.6 Canonical construction of pullbacks

In any category with all pullbacks, equalizers and products, there is a canonical way to construct pullbacks using products and equalizers. In this section we explain how this is done following ([1], Theorem 11.11). Let (Pb, α, β) be the pullback of the diagram (2.14)

$$A \xrightarrow{f} C \xleftarrow{g} B \tag{2.14}$$

First we construct the product $(A \times B, pr_A, pr_B)$ of A and B and then by the universal property of the product there exists a unique map $k : Pb \longrightarrow A \times B$ such that (2.15) is commutative.

$$\begin{tikzcd} & Pb \arrow[dl, bend right, "\alpha"'] \arrow[d, "k"] \arrow[dr, bend left, "\beta"] & \\ A & A \times B \arrow[l, "Pr_A"'] \arrow[r, "Pr_B"] & B \end{tikzcd} \tag{2.15}$$

Next, we construct the equalizer of the diagram (2.16)

$$\begin{tikzcd} A \times B \arrow[r, shift left, "fPr_A"] \arrow[r, shift right, "gPr_B"'] & C \end{tikzcd} \tag{2.16}$$

Suppose (E,e) is the equalizer of the diagram (2.16), then by the universal property of the equalizer there exists a unique map $u_1 : Pb \longrightarrow E$ such that (2.17) is commutative

$$\begin{tikzcd} E \arrow[r, "e"] & A \times B \arrow[r, shift left, "fPr_A"] \arrow[r, shift right, "gPr_B"'] & C \\ Pb \arrow[u, dotted, "u_1"] \arrow[ur, "k"'] & & \end{tikzcd} \tag{2.17}$$

Since $E \xrightarrow{e} A \times B$ is a map into a product $A \times B$, we have the commutative square (2.18)

$$\begin{tikzcd} E \arrow[r, "Pr_B e"] \arrow[d, "Pr_A e"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{2.18}$$

but (Pb, α, β) is the pullback of the diagram (2.14), so there exists a unique map $u_2 : E \longrightarrow Pb$. The maps $u_2 : E \longrightarrow Pb$ (unique map into the pullback) and $u_1 : Pb \longrightarrow E$ (unique map into the equalizer) are both universal maps such that (2.19) is commutative.

$$\begin{tikzcd}
E \arrow[drr, bend left, "Pr_B e"] \arrow[dr, bend right, "u_2"'] \arrow[ddr, bend right, "Pr_A e"'] & & \\
& Pb \arrow[ul, bend right, "u_1"'] \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd} \tag{2.19}$$

Therefore the equalizer of the diagram (2.16) is also the pullback of (2.14). It should also be noted that since the pullback is given as an equalizer of the diagram (2.16), pullback is a subobject of the product $A \times B$ ([6], pg 105).

2.2 Simplicial and cosimplicial objects

Simplicial objects are generalizations of the geometric simplicial complexes used in algebraic topology. First, we define the category of ordinal numbers and then give a definition of simplicial objects.

Definition 2.2.1. *The category of ordinal numbers denoted by Δ has*

1. *objects, the ordered sets $[n] = \{0, 1, \ldots, n\}, n \geq 0$.*

2. *morphisms, the order preserving maps $f : [n] \longrightarrow [m]$ (that is those maps f such that whenever $x \leq y$ then $f(x) \leq f(y)$).*

Theorem 2.2.2. *Every map f, in the category Δ can be uniquely decomposed ([54], 8.1.2) as compositions of*

1. *injective order preserving maps* $\delta^i : [n-1] \longrightarrow [n]$, $0 \leq i \leq n$ *given by* $j \mapsto j$ *for* $j < i$ *and* $j \mapsto j+1$ *for* $j \geq i$

$$\{0, 1, \ldots, i-1, i, i+1, \ldots, n-1\} \mapsto \{0, 1, \ldots, i-1, i+1, i+2, \ldots, n\}$$

2. *surjective order preserving maps* $\sigma^i : [n+1] \longrightarrow [n]$, $0 \leq j \leq n$ *given by* $j \mapsto j$ *for* $j \leq i$ *and* $j \mapsto j-1$ *for* $j > i$

$$\{0, 1, \ldots, i-1, i, i+1, i+2, \ldots, n+1\} \mapsto \{0, 1, \ldots, i-1, i, i, i+1, \ldots, n\}$$

.

Definition 2.2.3. *A **simplicial object** ([54], 8.1.3) S, in a category $\mathcal{C}$, is defined as a covariant functor $S : \Delta^{op} \longrightarrow \mathcal{C}$. Equivalently, a simplicial object in $\mathcal{C}$, is a sequence of objects A_n, $n \geq 0$ in $\mathcal{C}$, with face maps $d_i^n : A_n \longrightarrow A_{n-1}$, $0 \leq i \leq n$ and degeneracies $s_j^n : A_n \longrightarrow A_{n+1}$, $0 \leq j \leq n$, satisfying the simplicial identities ([54], 8.1.3):*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\ d_i s_j &= s_{j-1} d_i \quad \text{if } i < j \\ d_j s_j &= d_{j+1} s_j = id \\ d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1 \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j \end{aligned} \tag{2.20}$$

We can think of a simplicial object as a diagram

$$\begin{tikzcd} A_0 \arrow[r, "s_0^0"] & \; & A_1 \arrow[l, shift left, "d_0^1"'] \arrow[l, shift right, "d_1^1"] \arrow[r, shift left, "s_0^1"] \arrow[r, shift right, "s_1^1"'] & \; & A_2 \cdots \arrow[l, shift right=2, "d_0^2"'] \arrow[l, "d_1^2"] \arrow[l, shift left=2, "d_2^2"] \end{tikzcd} \tag{2.21}$$

Definition 2.2.4. *An augmented simplicial object ([54], 8.4.6) is a simplicial object $A_\bullet$ together with a map $\epsilon : A_0 \longrightarrow A_{-1}$, to an object $A_{-1} \in \mathcal{C}$ such that $\epsilon d_0 = \epsilon d_1$. The map $\epsilon : A_0 \longrightarrow A_{-1}$ is referred to as the augmentation.*

Definition 2.2.5. *A* ***cosimplicial object*** *([54], 8.1.4) cS, in a category $\mathcal{C}$, is defined as a covariant functor $cS : \Delta \longrightarrow \mathcal{C}$ or equivalently, as a sequence of objects A^n, $n \geq 0$ in $\mathcal{C}$, with coface maps $d_n^i : A^{n-1} \longrightarrow A^n$, $0 \leq i \leq n$ and degeneracies $s_n^j : A^{n+1} \longrightarrow A^n$, $0 \leq j \leq n$, satisfying the cosimplicial identities ([54], 8.1.4):*

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \quad \text{if } i < j \\ s^j d^i &= d^i s^{j-1} \quad \text{if } i < j \\ s^j d^j &= s^j d^{j+1} = id \\ s^j d^i &= d^{i-1} s^j \quad \text{if } i > j+1 \\ s^j s^i &= s^i s^{j+1} \quad \text{if } i \leq j \end{aligned} \tag{2.22}$$

Similar to a simplicial object we can represent a cosimplicial object as a diagram

$$\begin{tikzcd} A^0 \arrow[r, shift left=2, "d_1^0"] \arrow[r, shift right=2, "d_1^1"'] & A^1 \arrow[l, "s_0^0"'] \arrow[r, shift left=3, "d_2^0"] \arrow[r, "d_2^1"] \arrow[r, shift right=3, "d_2^2"'] & A^2 \cdots \arrow[l, shift right=2, "s_1^1"'] \arrow[l, shift left=2, "s_1^0"] \end{tikzcd} \tag{2.23}$$

Notation 2.2.6. *The category of simplicial objects over $\mathcal{C}$ is denoted $\mathfrak{s}\mathcal{C}$ and simplicial sets as $\mathfrak{s}\mathcal{SET}$. The category of cosimplicial objects over $\mathcal{C}$ is denoted $\mathfrak{c}\mathcal{C}$.*

Notation 2.2.7. *We will use Δ^n to denote the standard n-simplex and $\Delta^\bullet$ to denote the cosimplicial space with Δ^n in each cosimplicial dimension and the obvious coface and codegeneracy maps.*

Definition 2.2.8. *A map of simplicial objects $S \longrightarrow S'$ is a natural transformation of functors of the form $\Delta^{op} \longrightarrow \mathcal{C}$ (and dually maps of cosimplicial objects are natural transformations). Alternatively a simplicial map ([52], 2.1) $f : K \longrightarrow L$ between two simplicial objects K and L, in a category $\mathcal{C}$ is a family of morphisms $\{f_n\}_{n\geq 0}$ where $f_n : K_n \longrightarrow L_n$ satisfying*

1. $d_i f_n = f_{n-1} d_i$

2. $s_i f_n = f_{n+1} s_i \;\; 0 \leq i \leq n$.

2.2.1 Homotopy between two simplicial objects in a general category

We will use the following definition of a simplicial homotopy ([54], 8.3.11 or [45]) in the proof of Lemma 5.2.9 to show a simplicial homotopy between two maps of simplicial abelian groups.

Definition 2.2.9. *Let $X_\bullet$ and $Y_\bullet$ be simplicial objects in a category $\mathcal{C}$, and $f, g : X_\bullet \longrightarrow Y_\bullet$. A simplicial homotopy between f and g is a sequence of maps $h_i^n : X_n \longrightarrow Y_{n+1}$, for $0 \leq i \leq n$ and $n \geq 0$;*
such that

(A) $d_0^{n+1} h_0^n = g_n$

(B) $d_{n+1}^{n+1} h_n^n = f_n$

(C) $d_i h_j = h_{j-1} d_i \ \ (i < j)$

(D) $d_{j+1} h_{j+1} = d_{j+1} h_j$

(E) $d_i h_j = h_j d_{i-1} \ \ (i > j+1)$

(F) $s_i h_j = h_{j+1} s_i \ \ (i \leq j)$

(G) $s_i h_j = h_j s_{i-1} \ \ (i > j)$

2.2.2 Monads and comonads

Definition 2.2.10. *A **comonad** ([41], VI.1) in a category $\mathcal{C}$ consists of a functor $\perp : \mathcal{C} \longrightarrow \mathcal{C}$ and natural transformations,*

$$\epsilon : \perp \longrightarrow I \, \textit{(the counit) and} \quad \nu : \perp \longrightarrow \perp^2 \textit{(the comultiplication)}$$

such that the following diagrams commute.

$$\begin{tikzcd} \perp \arrow[r, "\nu"] \arrow[d, "\nu"'] & \perp^2 \arrow[d, "\perp\nu"] \\ \perp^2 \arrow[r, "\nu\perp"'] & \perp^3 \end{tikzcd} \tag{2.24}$$

$$\begin{tikzcd} & \perp \arrow[dl, equal] \arrow[d, "\nu"] \arrow[dr, equal] & \\ I\perp & \perp^2 \arrow[l, "\epsilon\perp"] \arrow[r, "\perp\epsilon"'] & \perp I \end{tikzcd} \tag{2.25}$$

There is a dual construction to comonads called monads.

Definition 2.2.11. *A* ***monad*** *([41], VI.1) in a category $\mathcal{C}$ consists of a functor $\top : \mathcal{C} \longrightarrow \mathcal{C}$ and natural transformations,*

$$\eta : I \longrightarrow \top \, \textit{(the unit) and} \quad \mu : \top^2 \longrightarrow \top \, \textit{(the multiplication)}$$

such that the following diagrams commute.

$$\begin{tikzcd} \top^3 \arrow[r, "\top\mu"] \arrow[d, "\mu\top"'] & \top^2 \arrow[d, "\mu"] \\ \top^2 \arrow[r, "\mu"'] & \top \end{tikzcd} \tag{2.26}$$

$$\begin{tikzcd} I\top \arrow[r, "\eta\top"] \arrow[dr, equal] & \top^2 \arrow[d, "\mu"] & \top I \arrow[l, "\top\eta"'] \arrow[dl, equal] \\ & \top & \end{tikzcd} \tag{2.27}$$

Theorem 2.2.12. *([26], ∮ 2.1)*

Every adjoint pair $F \dashv U$ with $U : \mathcal{D} \longrightarrow \mathcal{C}$, $F : \mathcal{C} \longrightarrow \mathcal{D}$, with unit of adjunction

$\eta : I_{\mathcal{C}} \longrightarrow UF$ and counit of adjunction $\epsilon : FU \longrightarrow I_{\mathcal{D}}$ gives rise to a comonad $(\perp, \epsilon, \nu)$ on $\mathcal{D}$, where

$$\begin{aligned} \perp &= FU : \mathcal{D} \longrightarrow \mathcal{D} \\ \epsilon &: \perp \longrightarrow I \\ \nu &= F\eta U : \perp \longrightarrow \perp^2 \end{aligned}$$

Dually an adjoint pair also gives rise to a monad ([26], $\oint$ 2.1), but we will not use this result and so we leave the details to the reader.

2.2.3 Construction of simplicial objects from a comonad

In [35], Huber shows that given a comonad $(\perp, \epsilon, \nu)$ on $\mathcal{C}$ and A an object in the category $\mathcal{C}$ we can define, for all $n \geq 0$,

$$\begin{aligned} A_n &: = \perp^{n+1} A \\ d_i^n &: \perp^{n+1} A \longrightarrow \perp^n A \ \text{ where, } \ d_i^n := \perp^i \epsilon \perp^{n-i} \\ s_i^n &: \perp^{n+1} A \longrightarrow \perp^{n+2} A \ \text{ where, } \ s_i^n := \perp^i \nu \perp^{n-i} \end{aligned} \tag{2.28}$$

to get a simplicial object in $\mathcal{C}$. The augmentation is given by $d_0^0 = \epsilon : \perp A \longrightarrow A$ (c.f. Definition 2.2.4).

Lemma 2.2.13 shows that for a comonad $(\perp, \epsilon, \nu)$ on $\mathcal{C}$, ϵ determines the augmentation.

Lemma 2.2.13. *The natural transformation ϵ of a comonad $(\perp, \epsilon, \nu)$ on $\mathcal{C}$ satisfies*

$$\epsilon(\epsilon\perp) = \epsilon(\perp\epsilon) \tag{2.29}$$

Proof. Let $L = \perp$, $R = I$, $M = \perp$, $N = I$, $\phi = \epsilon$, $\psi = \epsilon$ in equation (2.6), then we get diagram (2.30). The identity (2.29) holds in $\mathcal{C}$ is equivalent to diagram (2.30) being commutative

$$\begin{tikzcd} \perp\perp \arrow[r, "\perp\epsilon"] \arrow[d, "\epsilon\perp"'] & \perp I \arrow[d, "\epsilon I"] \\ I\perp \arrow[r, "I\epsilon"'] & II \end{tikzcd} \tag{2.30}$$

□

Dually, given a monad $(\top, \eta, \mu)$ on a category $\mathcal{C}$ with an object A, we can define $A^n = \top^{n+1}A$. The coface map $d_n^i : A^{n-1} \longrightarrow A^n$ is given by $d_n^i = \top^i\eta\top^{n-i}$ and the codegeneracy map $s_n^i : A^{n+1} \longrightarrow A^n$ is given by $s_n^i = \top^i\mu\top^{n-i}$.

Remark 2.2.14. *A dual result to identity (2.29) holds for a monad $(\top, \eta, \mu)$ in any category, which would imply $(\eta\top)\eta = (\top\eta)\eta$, thus giving a coaugmentation.*

2.2.4 Homotopy groups of a simplicial (abelian) group

Definition 2.2.15. *A* ***chain complex*** (C, ∂)*of groups is a sequence of groups and group homomorphisms*

$$C_0 \xleftarrow{\partial_1} C_1 \cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots \tag{2.31}$$

such that $Im(\partial_{n+1}) \subset ker(\partial_n)$, that is the composite $\partial_{n+1}\partial_n = 0$ for all n.

We can define the homotopy groups of a simplicial abelian group via the Moore chain complex. Let $G_\bullet \xrightarrow{\epsilon} G_{-1}$ be an augmented simplicial abelian group

$$\begin{tikzcd} G_{-1} & G_0 \arrow[l, "\epsilon"'] \arrow[r, "s_0^0"] & {} & G_1 \arrow[l, shift right, "d_1^1"'] \arrow[l, shift left, "d_0^1"] \arrow[r, shift left, "s_1^1"] \arrow[r, shift right, "s_0^1"'] & {} & G_2 \cdots \arrow[l, shift right=2, "d_2^2"'] \arrow[l, "d_1^2"] \arrow[l, shift left=2, "d_0^2"] \end{tikzcd} \tag{2.32}$$

Definition 2.2.16. *For the augmented simplicial abelian group (2.32) the associated Moore chain complex ([54], 8.4.6) is defined as*

$$\begin{aligned}
N_pG: &= G_p \quad \text{for } p \geq 0 \\
N_pG: &= G_{-1} \quad \text{for } p = -1 \\
N_pG: &= 0 \quad \text{for } p < -1 \\
\partial_p: &= \sum_{i=0}^{p} (-1)^i d_i^p \ \text{ where } p \geq 1 \\
\partial_0: &= \epsilon
\end{aligned} \tag{2.33}$$

The corresponding Moore chain complex for the diagram (2.32) is

$$0 \longleftarrow G_{-1} \xleftarrow{\epsilon} G_0 \xleftarrow{d_0 - d_1} G_1 \xleftarrow{d_0 - d_1 + d_2} G_2 \cdots$$

The addition of maps $d_0^p(x), \cdots, d_p^p(x)$ at a point $x \in G_p$, for the Moore chain complex (2.33) is defined as pointwise addition within the abelian group G_p.

Using Dold-Kan ([54], 8.4.1) correspondence we have the isomorphisms

$$\pi_n(G_\bullet) \cong H_n(N_pG)$$

.

Definition 2.2.17. *An augmented simplicial abelian group (2.32) is a simplicial resolution ([54], 8.4.6) of G_{-1} if*

(i) $\pi_n(G_\bullet) = 0$ *for* $n \geq 0$ *(acyclic)*

(ii) $\pi_0(G_\bullet) \cong G_{-1}$ *for* $n = 0$ *(augmentation gives an isomorphism)*

Fact 2.2.18. ([54], 8.4.6) In an abelian category the two conditions given in Definition 2.2.17 is equivalent to the augmented Moore chain complex being exact.

Let $G_\bullet$ be a simplicial group given by the diagram (2.34)

$$
\begin{tikzcd}
G_0 \arrow[r, "s_0^0"] & {} & G_1 \arrow[ll, shift left, "d_1^1"'] \arrow[ll, shift right, "d_0^1"] \arrow[r, shift left, "s_1^1"] \arrow[r, shift right, "s_0^1"'] & {} & G_2 \cdots \arrow[ll, shift left=2, "d_2^2"'] \arrow[ll, "d_1^2"] \arrow[ll, shift right=2, "d_0^2"]
\end{tikzcd}
\tag{2.34}
$$

Definition 2.2.19. *Moore chain complex* $(N_pG, \partial)_{p\geq 0}$ *of the simplicial group* $G_\bullet$ *is defined as ([42], 17.3)*

$$
\begin{aligned}
N_pG: &= G_0 \quad \text{for } p = 0 \\
N_pG: &= G_p \cap \ker d_1^p \cdots \cap \ker d_p^p \quad \text{for } p \geq 0 \\
\partial_p: &= d_0^p \ \text{ where } p \geq 0
\end{aligned}
$$

Using the Moore chain complex of the simplicial group $G_\bullet$, we can define the homotopy groups of the simplicial group $G_\bullet$ as ([42], 17.3, 17.4)

$$\pi_n(G_\bullet) = H_n(N_pG) = ker\ \partial_n / Im\ \partial_{n+1}$$

.

Fact 2.2.20. The homotopy groups of Definition 2.2.19 are isomorphic to those of Definition 2.2.16 if $G_\bullet$ is a simplicial abelian group.

Chapter 3

$\mathcal{H}(\mathcal{R})$-algebras

We will use spaces to mean pointed connected $\mathcal{CW}$-complexes and denote it by $\mathcal{CW}_*$. In this chapter we introduce objects that model the primary cohomology structure of a space with coefficients in a ring $\mathcal{R}$. We call these objects $\mathcal{H}(\mathcal{R})$-algebras and they are Eckmann-Hilton dual to the Π-algebras of homotopy theory.

First we introduce an $\mathcal{H}_N(\mathcal{R})$-algebra in Section 3.1, which is the algebraic structure of graded cohomology groups with all n-ary cohomology operations acting on them. In Section 3.2, we explain the free functor from the category of pointed sets to the category of $\mathcal{H}_N(\mathcal{R})$-algebras and also the unit and counit of adjunction. The functor description of $\mathcal{H}(\mathcal{R})$-algebras forms a model of a sketch in the sense of Ehresmann [25, 24] and this will be shown in Section 3.4.

In this thesis we will be interested in the cases when $\mathcal{R} = \mathbb{Z}$ or when $\mathcal{R} = \mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field with p elements and p is a prime number.

Definition 3.0.21. *An Eilenberg-Mac Lane space ([5], 2.5) over a ring $\mathcal{R}$ is a* *space denoted by $K(\mathcal{R}, n)$ with the property*

$$\pi_i(K(\mathcal{R}, n)) = \begin{cases} \mathcal{R} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Notation 3.0.22. *We will use $[A, B]$ to denote homotopy classes of pointed maps from A to B and as we will be working over any fixed ring $\mathcal{R}$, K^n will be used to mean $K(\mathcal{R}, n)$. The reduced cohomology groups of a space X are given by $H^n(X; \mathcal{R}) \cong [X, K^n]$ and we will use $H^n(X)$ to denote $H^n(X; \mathcal{R})$ because we will be working over any fixed ring $\mathcal{R}$.*

Definition 3.0.23. *$X \in \mathcal{C}$ is a group object ([31], $\oint$ 1) if and only if $[A, X]$ has a natural group structure for all $A \in \mathcal{C}$. Then $A \mapsto [A, X]$ gives a functor $\mathcal{CW}_*^{op} \longrightarrow \mathfrak{G}rp$.*

Fact 3.0.24. $\Omega K(\mathcal{R}, n) \cong K(\mathcal{R}, n-1)$ ([5], 2.5)

3.1 $\mathcal{H}_N(\mathcal{R})$-algebras

Definition 3.1.1. *Let $\mathcal{H}_N(\mathcal{R})$ be the category with objects, the finite products of Eilenberg-Mac Lane spaces over $\mathcal{R}$, including the point and morphisms the homotopy classes of maps between them.*

3.1.1 $\mathcal{H}_N(\mathcal{R})$-algebras as graded groups with operations

For any spaces X and Y and any map $f : X \longrightarrow Y$, for any homotopy class of maps $y \in H^n(Y)$ there is an induced abelian group homomorphism of cohomology $f^* : H^n(Y) \longrightarrow H^n(X)$ given by $f^*(y) = yf$. Primary operations give additional algebraic structure to the graded cohomology groups of a space which is natural, meaning that the morphism induced on cohomology by $f : X \longrightarrow Y$ respects this additional algebraic structure. This is formalized in the following definition.

Definition 3.1.2. *An n-ary cohomology operation $\theta : H^{m_1}(X)\times\cdots\times H^{m_n}(X) \longrightarrow H^q(X), n \in N$, is primary if, given any any spaces X and Y and any map $f : X \longrightarrow Y$, the following naturality diagram commutes.*

$$\begin{tikzcd} H^{m_1}(X) \times \cdots \times H^{m_n}(X) \arrow[r, "\theta"] & H^q(X) \\ H^{m_1}(Y) \times \cdots \times H^{m_n}(Y) \arrow[u, "f^*\times\cdots f^*"] \arrow[r, "\theta"] & H^q(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

Definition 3.1.3. *The cohomology $\mathcal{H}_N(\mathcal{R})$-algebra of a space X can be defined as the collection of graded abelian groups $\{H^n(X;\mathcal{R})\}_{n\in\mathbb{N}}$ with all the n-ary primary cohomology operations acting on these groups. Equivalently as a functor $[X,\ \ ] : \mathcal{H}(\mathcal{R}) \longrightarrow \mathcal{SET}_*$.*

Remark 3.1.4. *In Defintion 3.1.3 the $\mathcal{H}_N(\mathcal{R})$-algebra, satisfies what Blanc and Stover calls a 'category of universal graded algebras' (CUGA) [9]. All CUGA's have all limits and colimits [9].*

There is another definition of $\mathcal{H}_N(\mathcal{R})$-algebras, which is more general than Definition 3.1.3. The second definition describes $\mathcal{H}_N(\mathcal{R})$-algebras as a functor and this definition is valid for abstract $\mathcal{H}_N(\mathcal{R})$-algebras that do not come from a space.

3.1.2 $\mathcal{H}_N(\mathcal{R})$-algebras as a functor

Definition 3.1.5. *A $\mathcal{H}_N(\mathcal{R})$-algebra is a functor from $\mathcal{H}_N(\mathcal{R})$ to the category $\mathcal{SET}_*$ of graded pointed sets, preserving products and sending the point to 0.*

In Figure 3.1, we have denoted $Z(K^i)$ as Z^i and $Z(*)$ as 0. All identical copies of Z^i are identified in $\mathcal{SET}_*$.

The maps of $\mathcal{H}_N(\mathcal{R})$ induce primary operations on the graded set Z^i, $i \in I$ including the abelian group addition.

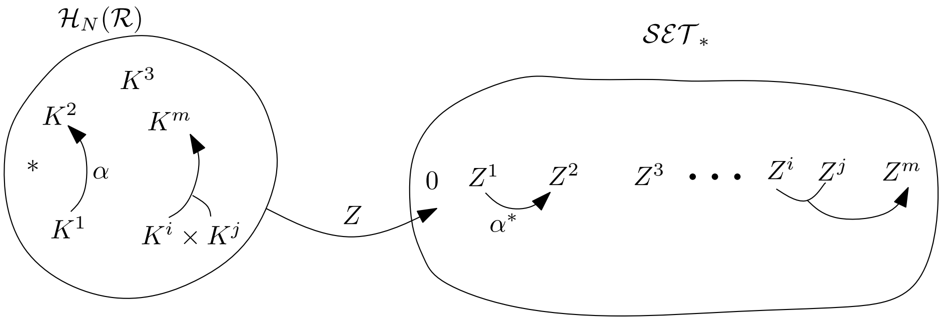

Figure 3.1

Example 3.1.6. For any $X \in \mathcal{CW}_*$,

$$[X,\ \ ] : \mathcal{H}_N(\mathcal{R}) \longrightarrow \mathcal{SET}_*$$

is an $\mathcal{H}_N(\mathcal{R})$-algebra with the image the cohomology $\mathcal{H}_N(\mathcal{R})$-algebra given in Definition 3.1.3

From Definition 3.1.5 we know that a $\mathcal{H}_N(\mathcal{R})$-algebra morphism is a natural transformation.

Definition 3.1.7. *A $\mathcal{H}_N(\mathcal{R})$-algebra morphism $\lambda : Z \longrightarrow W$ is a natural transformation. That is, for every $C \in \mathcal{H}_N(\mathcal{R})$ there exists $\lambda_C : Z(C) \longrightarrow W(C)$ such that for any morphism $\alpha : C \longrightarrow C'$ in $\mathcal{H}_N(\mathcal{R})$ we have $W(\alpha)\lambda_C = \lambda_{C'}Z(\alpha)$.*

Notation 3.1.8. *The category of $\mathcal{H}_N(\mathcal{R})$-algebras and $\mathcal{H}_N(\mathcal{R})$-algebra morphisms will be denoted by $\mathcal{H}_N(\mathcal{R}) - \mathcal{ALG}$.*

3.2 Free $\mathcal{H}(\mathcal{R})$-algebras

The cohomology $\mathcal{H}_N(\mathcal{R})$-algebra of a product of Eilenberg-Mac Lane spaces is a free $\mathcal{H}_N(\mathcal{R})$-algebra. This can be shown theoretically using the Yoneda Lemma as in [12]. In [46], Percy showed an explicit construction of a free functor which is left adjoint to the underlying functor between $\mathcal{H}_N(\mathcal{R}) - \mathcal{ALG}$ and $\mathcal{SET}_*$. In this section we will explain the free functor F, given in [46] followed by the unit and counit of adjunction.

3.2.1 Free functor F

Let $G_\bullet$ and $B_\bullet$ be graded pointed sets and $f : G_\bullet \longrightarrow B_\bullet$ be a graded function, where we denote $f_n : G_n \longrightarrow B_n$ to be a function on each grade. Let F be a functor from the category of graded pointed sets to $\mathcal{H}_N(\mathcal{R}) - \mathcal{ALG}$. We define

$$FG_\bullet := [\prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n, \quad] \quad \text{where } G_n^- = G_n \setminus *$$

To show how F acts on the map f, let $j \in J_b$ index the set of points in G_n^- whose image under f_n is $b \in B_n^-$. That is, $f_n(g_j) = b$, for some $b \in B_n^-$. We identify the factors $id_{K^n} : K_b^n \longrightarrow K_{g_j}^n$ and using the universal property of the product we get the map $\{id_{K^n}\}_{j\in J_b} : K_b^n \longrightarrow \prod_{n\in\mathbb{N}} \prod_{j\in J_b} K_{g_j}^n$ in diagram (3.1).

$$\begin{tikzcd}
 & & K_{g_j}^n & \\
K_b^n \arrow[urr, "id_{K^n}"] \arrow[rr, "\{id_{K^n}\}_{j\in J_b}"'] & & \prod_{n\in\mathbb{N}} \prod_{j\in J_b} K_{g_j}^n \arrow[u, "pr_{g_j}"'] & \subset \prod_{n\in\mathbb{N}} \prod_{g\in G_n} K_g^n
\end{tikzcd} \tag{3.1}$$

Define $\hat{f} : \prod_{n\in\mathbb{N}} \prod_{b\in B_n^-} K_b^n \xrightarrow{\text{project}} \prod_{n\in\mathbb{N}} \prod_{b\in Imf_n^-} K_b^n \xrightarrow{\prod_{n\in\mathbb{N}} \prod_{b\in Imf_n^-} \{1_{K^n}\}_{j\in J_b}} \prod_{n\in\mathbb{N}} \prod_{g\in G_n} K_g^n$

then we get induced map $\hat{f}^* : [\prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n, \quad] \longrightarrow [\prod_{n\in\mathbb{N}} \prod_{b\in B_n^-} K_b^n, \quad]$.

We define $Ff = \hat{f}^* : FG_\bullet \longrightarrow FB_\bullet$.

3.2.2 Unit of adjunction

The unit of adjunction is defined by the natural transformation $\eta_{G_\bullet} : G_\bullet \longrightarrow UFG_\bullet$, where $g \in G_n^-$ is taken to $pr_g : \prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n \longrightarrow K_g^n$ and $*$ is taken to $0 \in H^n(\prod_{n\in\mathbb{N}} \prod_{g\in G_n^-} K_g^n)$.

3.2.3 Counit of adjunction

Denote $Z^{n-} = Z(K^n)^-$ to be the image under the functor Z of K^n in $\mathcal{SET}_*$, without the basepoint. Form the product $\prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} Z_u^{n-}$.

Select the element $(u)_{u\in Z^{n-},n\in\mathbb{N}}$ that takes the element u from the set Z_u^{n-} indexed by u. Then we can identify

$$\begin{aligned}\prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} Z_u^{n-} &\equiv \prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} Z(K^n)_u \\ &\equiv Z(\prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} K_u^n)\,,\text{ since } Z \text{ preserves products.}\end{aligned}$$

So we identify $(u)_{u\in Z^{n-},n\in\mathbb{N}}$ with $w \in Z(\prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} K_u^n)$.

Let $\alpha \in FUZ$ be given by a cohomology class of the map $\alpha : \prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} K_u^n \longrightarrow K^p$ for some $p \in \mathbb{N}$, then $Z\alpha$ is an operation $Z(\prod_{n\in\mathbb{N}} \prod_{u\in Z^{n-}} K_u^n) \longrightarrow Z(K^p)$.

For every $p \in \mathbb{N}$, we define the counit of adjunction $\epsilon_Z : FUZ(K^p) \longrightarrow Z(K^p)$ by $\alpha \mapsto Z\alpha(w) \in Z^p$.

3.3 $\mathcal{H}(\mathcal{R})$-algebra

In Chapter 6, we give a construction $T(X)$ on a space X, which is Eckmann-Hilton dual to Stover's construction $V(X)$ from ([53], 2.2). The construction $T(X)$ is homotopy equivalent to an infinite product of Eilenberg-Mac Lane spaces over $\mathcal{R}$.

For a cohomology $\mathcal{H}_N(\mathcal{R})$-algebra of a product of Eilenberg-Mac Lane spaces to be free, the map representing $\alpha \in FUZ$ needs to be contained in $\mathcal{H}_N(\mathcal{R})$, otherwise the counit of adjunction is not defined. Thus we need to modify our definition of $\mathcal{H}_N(\mathcal{R})$ to contain universal arrows for infinitary operations.

In the case of $\mathcal{R} = \mathbb{F}_p$, all maps out of an infinite product of Eilenberg-Mac Lane spaces over $\mathbb{F}_p$ factor through a finite subproduct because $\mathbb{F}_p$ is algebraically compact ([34], Prop 2.1). So $\mathcal{H}(\mathbb{F}_p)$ only needs to contain finite products to define free $\mathcal{H}(\mathbb{F}_p)$-algebras.

However, for a general ring $\mathcal{R}$ and $\mathcal{R} = \mathbb{Z}$ in particular, the property that maps out of an infinite product of Eilenberg-Mac Lane spaces do not factor through a finite subproduct because $\mathbb{Z}$ is not algebraically compact. So to define free $\mathcal{H}_N(\mathbb{Z})$-algebras we need the category $\mathcal{H}_N(\mathbb{Z})$ to contain universal arrows for infinitary operations. Thus we have the definition;

Definition 3.3.1. *Let $\mathcal{H}(\mathcal{R})$ be the category with objects, arbitrary (possibly infinite) products of Eilenberg-Mac Lane spaces over $\mathcal{R}$, including the point and morphisms the homotopy classes of maps between them.*

Definition 3.3.2. *An $\mathcal{H}(\mathcal{R})$-algebra is a functor from $\mathcal{H}(\mathcal{R})$ to $\mathcal{SET}_*$, preserving products and sending the point to 0.*

Definition 3.3.3. *The cohomology $\mathcal{H}(\mathcal{R})$-algebra of a space can be defined as the collection of graded abelian groups $\{H^n(X; \mathcal{R})\}_{n\in\mathbb{N}}$ with all the primary cohomology operations acting on these groups.*

Notation 3.3.4. *The category of $\mathcal{H}(\mathcal{R})$-algebras and $\mathcal{H}(\mathcal{R})$-algebra morphisms will be denoted by $\mathcal{H}(\mathcal{R}) - \mathcal{ALG}$.*

3.4 Sketches

Lawvere introduced, in his 1963 doctorial thesis [39], an alternative method of encoding algebraic theories without using generators and their relations. Historically the problem of the structure of cohomology algebras has been studied from the perspective of a universal graded algebra. These rely on knowing the generators of the primary operations and their relations. Since the generators and their relations of $\mathcal{H}(\mathcal{R})$-algebras are not always known for a general ring $\mathcal{R}$, in this thesis we will apply the technique of Lawvere theories in which the generators and their relations are encoded in a category and do not need to be known explicitly. The idea behind Lawvere theory is that morphisms in a Lawvere theory correspond to the operations of the algebraic theory. The relations satisfied by the generators correspond to the the fact that certain morphisms are equivalent. We will use a generalization of Lawvere theory called models of sketches introduced by Ehresmann in [25, 24]. The functor definition of $\mathcal{H}(\mathcal{R})$-algebras allows us to define the $\mathcal{H}(\mathcal{R})$-algebras in the context of Ehresmann's models of sketches.

Definition 3.4.1. *A sketch $\mathbb{S} = \{\mathcal{T}, \mathcal{P}, \mathcal{I}\}$ ([13], 5.6.1) is a triple with a small pointed category $\mathcal{T}$ with*

(i) a set $\mathcal{P}$ of cones on functors $R : \mathcal{D} \longrightarrow \mathcal{T}$, defined on small categories $\mathcal{D}$;

(ii) a set $\mathcal{I}$ of cocones on functors $R : \mathcal{D} \longrightarrow \mathcal{T}$, defined on small categories $\mathcal{D}$.

Definition 3.4.2. *Let $\mathbb{S}$ be a sketch. A model ([13], 5.6.2) of a sketch $\mathbb{S}$ in a category $\mathcal{W}$ is a covariant functor $\mathbb{S}$-model $\mathcal{W} : \mathcal{T} \longrightarrow \mathcal{W}$ which preserves limits and colimits of the category $\mathcal{T}$.*

Remark 3.4.3. *Definitions 3.4.1 and 3.4.2 are too general for our purposes and we will restrict the set $\mathcal{P}$ in Definition 3.4.1 to contain only small products, since* *we are not interested in other limits and the set $\mathcal{I}$ to be empty. Therefore, by a model of a sketch in Definition 3.4.2 we mean only product preserving functors.*

Definition 3.4.4. *If $\mathcal{P}$ in Definition 3.4.1 contains only finite products then the sketch is referred to as a finite product sketch.*

Remark 3.4.5. *For our purposes we take $\mathcal{T}$-models in $\mathcal{SET}_*$. That is functors of the following form $W : \mathcal{T} \longrightarrow \mathcal{SET}_*$, and homomorphisms of $\mathcal{T}$-models are natural transformations between such functors.*

Definition 3.4.6. *A sketch $\mathcal{T}$ is **multisorted** with objects from a set of sorts S, if every object in $\mathcal{T}$ is a product of elements from S.*

Example 3.4.7. Let Δ be the category of finite ordinals and order preserving maps. Setting $\mathcal{T} = \Delta^{op}$, the $\mathcal{T}$-models in a category $\mathcal{C}$ are the simplicial objects in $\mathcal{C}$. Similarly the Δ-models in a category $\mathcal{C}$ form the cosimplicial objects of $\mathcal{C}$. Both of these theories $\mathcal{T} = \Delta^{op}$ and $\mathcal{T} = \Delta$ are single sorted, with the set of all the objects of Δ thought of as the sort.

Example 3.4.8. The category of $\mathcal{H}_N(\mathcal{R})$-algebras and $\mathcal{H}_N(\mathcal{R})$-algebra morphisms given in Definition 3.1.5 and 3.1.7 is corepresentable by $\mathcal{T} = \mathcal{H}_N(\mathcal{R})$. $\mathcal{H}_N(\mathcal{R})$ is sorted by the spaces $K(\mathcal{R}, n)$, $n \geq 1$. Then $\mathcal{H}_N(\mathcal{R})$-algebras are the $\mathcal{H}_N(\mathcal{R})$-models in $\mathcal{SET}_*$. This agrees with the Definition 3.1.5 of $\mathcal{H}_N(\mathcal{R})$-algebras being the product preserving functors from $\mathcal{H}_N(\mathcal{R})$ to $\mathcal{SET}_*$.

Similarly, if we let $\mathcal{T} = \mathcal{H}(\mathcal{R})$, then the $\mathcal{H}(\mathcal{R})$-models in $\mathcal{SET}_*$ are $\mathcal{H}(\mathcal{R})$-algebras given in Definition 3.3.2.

Example 3.4.9. Let $\mathcal{H}(\mathbb{F}_p)$ be the category of Definition 3.3.1 with $\mathcal{R} = \mathbb{F}_p$. Then $\mathcal{H}(\mathbb{F}_p)$ is sorted by the spaces $K(\mathbb{F}_p, n)$, $n \geq 1$ and the models of $\mathcal{H}(\mathbb{F}_p)$ in $\mathcal{SET}_*$ are algebras over the mod-p Steenrod algebra ([8], $\oint 1$).

Remark 3.4.10. *All the finite product sketches modelled on $\mathcal{SET}_*$ have all limits and colimits [2]. In [40], Linton gives a general proof for the existence of a canonical free functor F from $\mathcal{SET}$ to the models of a finite product sketch $\mathcal{T}$-model, that is left adjoint to the underlying functor U.*

Remark 3.4.11. *In ([13] 5.6.8) it is shown for a sketch $\mathbb{S} = \{\mathcal{T}, \mathcal{P}, \mathcal{I}\}$ if either $\mathcal{P}$ or $\mathcal{I}$ is empty then the models of $\mathbb{S}$ are locally presentable. Adámek and Rosicky ([2] or [13] 5.5.8), show that all locally presentable categories have all limits and colimits. Therefore, even in the general case with infinite product of Eilenberg-Mac Lane spaces, for the category of $\mathcal{H}(\mathcal{R})$-models in $\mathcal{SET}_*$ all limits and colimits exist because $\mathcal{I}$ is empty (c.f. Remark 3.4.3).*

3.5 Simplicial $\mathcal{H}(\mathcal{R})$-algebras

The free and underlying adjunction between $\mathcal{H}(\mathcal{R}) - \mathcal{ALG}$ and $\mathcal{SET}_*$ gives rise to comonad on $\mathcal{H}(\mathcal{R}) - \mathcal{ALG}$ by Theorem 2.2.12. Using this comonad we can form an augmented simplicial $\mathcal{H}(\mathcal{R})$-algebra for an $\mathcal{H}(\mathcal{R})$-algebra Z as explained in Section 2.2.3.

$$Z \xleftarrow{d_0^0} FU(Z) \xrightarrow{s_0^0} \overset{\xleftarrow{d_1^1}}{\underset{\xleftarrow{d_0^1}}{}} FU^2(Z) \overset{\xrightarrow{s_0^1}}{\underset{\xrightarrow{s_1^1}}{}} \overset{\xleftarrow{d_2^2}}{\underset{\xleftarrow{d_0^2}}{\xleftarrow{d_1^2}}} FU^3(Z) \cdots \tag{3.2}$$

Each $FU^n(Z)$, for $n > 0$ in (3.2) is a free $\mathcal{H}(\mathcal{R})$-algebra when considered individually.

In Figure 3.2, we have denoted $FU^n(Z)(K^i)$ as $FU^n(Z^i)$ and $Z(K^i)$ as (Z^i). Each row in Figure 3.2 is an $\mathcal{H}(\mathcal{R})$-algebra (c.f. Figure 3.1). All identical copies of $FU^n(Z^i)$ in $\mathcal{SET}_*$ are identified. As illustrated in Figure 3.2, cohomology operations on $FU^n(Z)$, for $n > 0$ is induced by the cohomology operations in Z, because FU^n is a functor.

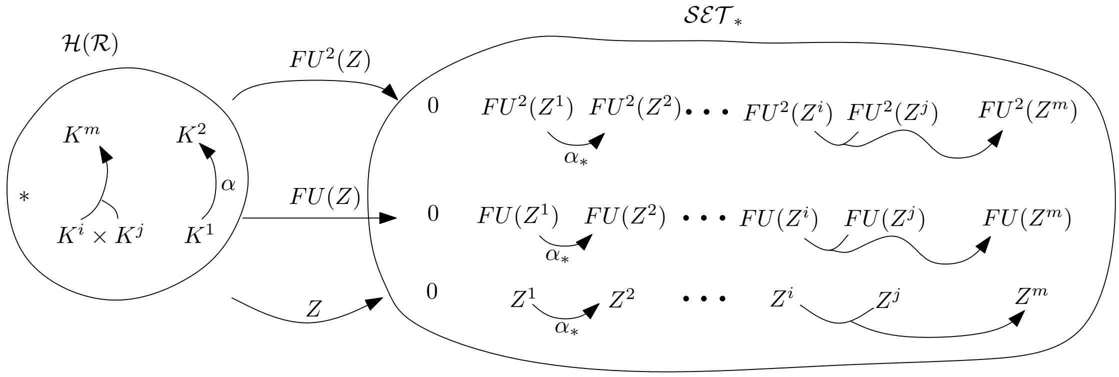

Figure 3.2

A simplicial $\mathcal{H}(\mathcal{R})$-algebra can be explained using Figure 3.3, where a horizontal strip shows a graded abelian group with primary cohomology operations acting on them (each row is an $\mathcal{H}(\mathcal{R})$-algebra). In Figure 3.3 any vertical strip gives a simplicial abelian group because the natural transformations d_i^n and s_i^n in (3.2) act on the j^{th} column to give natural transformations $d_{i\;Z^j}^n$ and $s_{i\;Z^j}^n$ and these natural transformations satisfy the simplicial identities.

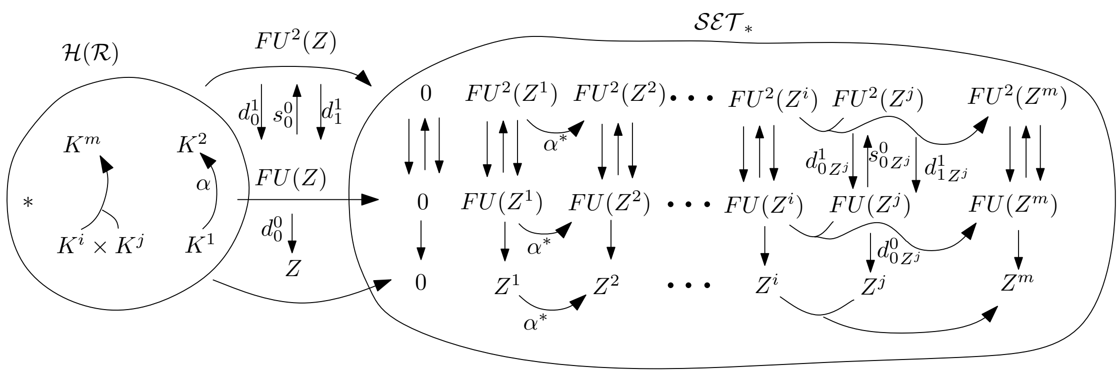

Figure 3.3

Notation 3.5.1. *We will use $(FU)_\bullet(Z)$ to denote the simplicial $\mathcal{H}(\mathcal{R})$-algebra on Z constructed using the free functor F and underlying functor U.*

Lemma 3.5.2. *Let $(FU)_\bullet(Z)$ be a simplicial $\mathcal{H}(\mathcal{R})$-algebra augmented by the $\mathcal{H}(\mathcal{R})$-algebra Z_{-1}. The simplicial maps d_i^n and s_i^n induce a morphism of $\mathcal{H}(\mathcal{R})$-algebras.*

Proof. The face and degeneracy maps in a simplicial $\mathcal{H}(\mathcal{R})$-algebra are the natural transformations d_i^n and s_i^n given in equation (2.28), therefore these natural transformations commute with all the primary cohomology operations. Hence morphisms of $\mathcal{H}(\mathcal{R})$-algebras. $\square$

Chapter 4

Model categories

To construct resolutions in Chapter 4 and Chapter 5, we need a proper framework to define resolutions and model category theory allows us to do this. The purpose of this chapter is to define all the model category structures that will be used in this thesis.

In the first section we give the model category axioms and in Section 4.1.1 we give a model category structure for a simplicial model of a finite product sketch. Then we describe Bousfield's resolution model category structure [15] on the category $c\mathcal{C}$ of cosimplicial objects over a model categroy $\mathcal{C}$. Bousfield's resolution model category of cosimplicial spaces is a generalization of the Dwyer-Kan-Stover [22] theory of resolution model category on simplicial spaces.

4.1 Model category structure

Model categories, developed by Quillen in [47] give the most general context in which the tools of homotopy theory can be used. Homotopy theory allows problems in a general category to be reformulated in a more tractable algebraic setting.

A model category is a category with three distinct classes of morphisms satisfying axioms making localization functorial with an image in $\mathfrak{G}rp$. Categories which satisfy the model category axioms include the category of chain complexes over a commutative ring, the category of topological spaces and a category of simplicial sets. The homotopy theory of the category of chain complexes forms what we know as homological algebra (abelian case), but, the main purpose of model category theory is to study homotopy theory in non-abelian categories of which topological spaces are a motivating example. The category of Π-algebras and also the category of $\mathcal{H}(\mathcal{R})$-algebras are non-abelian categories, so we need the notion of model category to do homotopy on these categories.
Before we define a model category we give the following definition [23]

Definition 4.1.1. *A map $g : A \longrightarrow B$ is a retract of a map $f : X \longrightarrow Y$ in a category, if there is a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"] & X \arrow[r, "r"] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r, "j"] & Y \arrow[r, "s"] & B \end{tikzcd}$$

with $ri = 1_A$ and $sj = 1_B$.

The reason why we defined retracts is because isomorphisms are closed under retracts (that is, the retract of an isomorphism is an isomorphism) in any category. In order to localize over weak equivalences so they become isomorphisms in the quotient category we will require the distinguished classes of maps of a model category to be closed under retracts.

Definition 4.1.2. *Let $\mathcal{C}$ be a category. A model category structure ([23], $\oint 3$) on $\mathcal{C}$ is given by 3 classes of maps : weak equivalences, fibrations and cofibrations, satisfying the axioms.*

(MC1) Finite limits and colimits exist in $\mathcal{C}$

(MC2) If f and g are maps in $\mathcal{C}$ such that gf is defined and if any two of f, g, gf is a weak equivalence then so is the third.

(MC3) Fibrations, cofibrations and weak equivalences are closed under retracts.

(MC4) Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where $i : A \longrightarrow B$ is a cofibration and $p : X \longrightarrow Y$ is a fibration, then a lift shown in the diagram on the right by the dotted arrow exists, if i or p is also a weak equivalence.

(MC5) Every map f in $\mathcal{C}$ can be factored as $f = pi$ in two ways, where

i is a cofibration and p is a fibration and a weak equivalence.

p is a fibration and i is a cofibration and a weak equivalence.

A map that is a fibration and a weak equivalence is called an **acyclic fibration** and a map that is a cofibration and a weak equivalence is called a **acyclic cofibration**. (MC1) implies that there is an initial object ϕ and a terminal object e in any model category. In a model category, A is a **cofibrant** object if there is a cofibration $\phi \longrightarrow A$, and B is a **fibrant** object if there is a fibration $B \longrightarrow e$.

Definition 4.1.3. *The **homotopy category** $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and morphisms between X and Y as $[\hat{\hat{X}}, \hat{\hat{Y}}]$ where $\hat{\hat{X}}$ and $\hat{\hat{Y}}$ denotes a cofibrant fibrant replacement for X and Y respectively.*

Remark 4.1.4. *The cofibrations in a model category are the maps that satisfy the Left Lifting Property (LLP) with repsect to the acyclic fibrations and the acyclic* *cofibrations are maps that satisfy the LLP with respect to the fibrations. The dual statements also hold. Using this criteria once a fibration (or a cofibration) is fixed then cofibrations (or a fibration) are already determined [23].*

4.1.1 Model category structure on simplicial models of a finite product sketch

One of the fundamental observations in Quillen's work on homotopical algebra [47], is that, in an non-abelian category, resolutions of an object by a chain complex has to be replaced by a simplicial resolution.

The following Theorem shows there is a canonical model category structure given on any simplicial model of a finite product sketch (c.f. remark 3.4.10). If $\mathcal{T}$ is a finite product sketch the category of simplicial $\mathcal{T}$-models will be denoted by $s\mathcal{T}$-model. This method was initially used by Quillen ([47], II.4) to show a model structure on varieties of algebras and later generalised by other authors [33, 32]. The idea behind it is to transport the model structure on $s\mathcal{SET}_*$ to one on $s\mathcal{T}$-model using the adjunction $T\text{-model} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{SET}_*$, where U and F are underlying and free functors respectively.

Fact 4.1.5. ([8], prop 4.1) Let $\mathcal{T}$ be a finite product sketch, then there is a model category structure on $s\mathcal{T}$-models. Let $X_\bullet, Y_\bullet \in s\mathcal{T}$-models. In this model structure a map $f : X_\bullet \longrightarrow Y_\bullet$ of $s\mathcal{T}$-model is

1. a weak-equivalence if $U(T)f : U(X_\bullet) \longrightarrow U(Y_\bullet)$ is weak-equivalence of $s\mathcal{SET}_*$.

2. a fibration if $U(T)f : U(X_\bullet) \longrightarrow U(Y_\bullet)$ is a Kan fibration.

The cofibrations are the maps that satisfy the left lifting property with repect to the fibrations.

Example 4.1.6. In Example 3.4.8, we showed $\mathcal{H}_N(\mathcal{R})$-algebras are an $\mathcal{H}_N(\mathcal{R})$-model in $\mathcal{SET}_*$. Then we can apply Theorem 4.1.5 to get a model category structure on simplicial $\mathcal{H}_N(\mathcal{R})$-algebras.

4.1.2 Resolution model structure on $s\mathcal{T}$-model

In this section we explain the model category stucture from [8], which can be applied to simplicial $\mathcal{H}(\mathcal{R})$-algebras from Section 3.5, for $\mathcal{R} = \mathbb{Z}$. This is very important because it is hard to define model category structures on simplicial models of infinite product sketches but if the models of a sketch are locally presentable then using Fact 4.1.8 we can give a resolution model category structure.

Definition 4.1.7. *([8], $\oint$ 4) Let $A_\bullet$ be a simplicial $\mathcal{T}$-model of a sketch in $\mathcal{SET}_*$. By $\pi_p A_\bullet$ we mean the composition*

$$\pi_p A_\bullet : \mathcal{T} \xrightarrow{A_\bullet} s\mathcal{SET}_* \xrightarrow{\pi_p} \mathcal{SET}_*$$

where π_p is the p^{th} homotopy group functor on simplicial sets.

Fact 4.1.8. ([8], $\oint$ 4) If a locally presentable model of a sketch $\mathcal{T}$-model (c.f. Remark 3.4.11) has an underlying graded group structure then a map of simplicial $\mathcal{T}$-models $f : W_\bullet \longrightarrow Z_\bullet$ is a

i. weak equivalence if and only if $\pi_n f : \pi_n W_\bullet \longrightarrow \pi_n Z_\bullet$ is an isomorphism for all $n \geq 0$.

ii. fibration if and only if the underlying map as graded groups is a surjection onto the base point component.

Cofibrations are the maps that satisfy the left lifting property with respect to the fibrations.

4.2 Model category on Cosimplicial objects

4.2.1 Latching and Matching objects for cosimplicial objects in a category

Definition 4.2.1. *Let $X^\bullet$ be a cosimplicial object. The latching objects $L^n X^\bullet$ in $\mathcal{C}$ for $n \geq 0$ ([15], 2.2), are defined as*

$$L^n X^\bullet = \operatorname{colim}_{\theta:[k]\to[n]} X^k$$

with θ ranging over the injections $[k] \to [n]$ in Δ for $k < n$. The maps $L^n X^\bullet \longrightarrow X^n$ in $\mathcal{C}$ are the latching maps. Similarly we define the matching objects ([15], 2.2) $M^n X^\bullet$ in $\mathcal{C}$ for $n > 0$ as

$$M^n X^\bullet = \lim_{\theta:[n]\to[k]} X^k$$

with θ ranging over surjections $[n] \to [k]$ in Δ for $k < n$. The maps $X^n \longrightarrow M^n X^\bullet$ are the matching maps.

4.2.2 Reedy model structure

Definition 4.2.2. *Let $\mathcal{C}$ be a model category. A map $f : X^\bullet \longrightarrow Y^\bullet$ in $c\mathcal{C}$ is called ([48] or [15], 2.2):*

i. a Reedy weak equivalence if each $f : X^n \longrightarrow Y^n$ is a weak homotopy equivalence in $\mathcal{C}$ for each $n \geq 0$;

ii. a Reedy fibration if the map $X^n \longrightarrow Y^n \times_{M^n Y^\bullet} M^n X^\bullet$ (c.f. Notation 2.1.24) is a fibration in $\mathcal{C}$ for all $n \geq 0$;

iii. a Reedy cofibration if the map $X^n \coprod_{L^n X^\bullet} L^n Y^\bullet \longrightarrow Y^n$ (c.f. Notation 2.1.26) is a cofibration in $\mathcal{C}$ for all $n \geq 0$.

Fact 4.2.3. ([48] or [15], 2.2) If $\mathcal{C}$ is a model category then $c\mathcal{C}$ has a model category structure called a Reedy model category, with Reedy weak equivalences, Reedy fibrations and Reedy cofibrations.

4.3 Resolution model category on spaces

4.3.1 $\mathcal{G}$-injectives

Definition 4.3.1. *A model category is called a left proper pointed model category ([15], 3.1), if each pushout of a weak equivalence along a cofibration is a weak equivalence.*

Definition 4.3.2. *([15], 3.1) Let $\mathcal{C}$ be a left proper pointed model category and $\mathcal{G}$ be a class of group objects in the homotopy category $Ho(\mathcal{C})$. A map $i : A \longrightarrow B$ in $Ho(\mathcal{C})$ is called $\mathcal{G}$-monic when $i^* : [B, \Omega^n G] \longrightarrow [A, \Omega^n G]$ is onto for each $G \in \mathcal{G}$ and $n \geq 0$. A map in $\mathcal{C}$ is called $\mathcal{G}$-monic when the induced map is $\mathcal{G}$-monic in $Ho(\mathcal{C})$.*

Definition 4.3.3. *([15], 3.1) An object $C \in Ho(\mathcal{C})$ is called $\mathcal{G}$-injective when $i^* : [B, \Omega^n C] \longrightarrow [A, \Omega^n C]$ is onto for each $\mathcal{G}$-monic map $i : A \longrightarrow B$ in $Ho(\mathcal{C})$ and $n \geq 0$. An object C in $\mathcal{C}$ is called $\mathcal{G}$-injective when it is $\mathcal{G}$-injective in $Ho(\mathcal{C})$.*

Definition 4.3.4. *([15], 3.1) $Ho(\mathcal{C})$ has enough $\mathcal{G}$-injectives when each object of $Ho(\mathcal{C})$ is the source of a $\mathcal{G}$-monic map to a $\mathcal{G}$-injective target. A class of group objects $\mathcal{G}$ in $Ho(\mathcal{C})$ is called a class of injective models if $Ho(\mathcal{C})$ has enough $\mathcal{G}$-injectives.*

4.3.2 $\mathcal{G}$-resolution model structure on $c\mathcal{C}$

The model category on simplicial groups is defined as follows.

Definition 4.3.5. *A homomorphism in the category $s\mathcal{G}rp$ is a weak equivalence or a fibration when its underlying map in $s\mathcal{SET}$ is a weak equivalence or a fibration. The cofibrations of $s\mathcal{G}rp$ are the maps that satisfy the left lifting property with respect to all the acyclic cofibrations. ([47], II.3).*

Definition 4.3.6. *A map $f : X^\bullet \longrightarrow Y^\bullet$ in $c\mathcal{C}$ is called*

1. *a $\mathcal{G}$-equivalence when $f^* : [Y^\bullet, \Omega^n G] \longrightarrow [X^\bullet, \Omega^n G]$ is a weak equivalence in $s\mathcal{G}rp$ for each $G \in \mathcal{G}$ and $n \geq 0$.*

2. *a $\mathcal{G}$-cofibration when f is a Reedy cofibration and $f^* : [Y^\bullet, \Omega^n G] \longrightarrow [X^\bullet, \Omega^n G]$ is a cofibration in $s\mathcal{G}rp$ for each $G \in \mathcal{G}$ and $n \geq 0$.*

3. *a $\mathcal{G}$-fibration when $f : X^n \longrightarrow Y^n \times_{M^n Y^\bullet} M^n X^\bullet$ is a $\mathcal{G}$-injective fibration in $\mathcal{C}$ for $n \geq 0$.*

Definition 4.3.7. *Let $c\mathcal{C}^{\mathcal{G}}$ denote the category $c\mathcal{C}$ with weak equivalences defined as $\mathcal{G}$-equivalences, with cofibrations as $\mathcal{G}$-cofibrations and fibrations as $\mathcal{G}$-fibrations.*

Fact 4.3.8. ([15], 3.3) If $\mathcal{C}$ is a left proper pointed model category with a class $\mathcal{G}$ of injective models in $Ho(\mathcal{C})$ then $c\mathcal{C}^{\mathcal{G}}$ is a left proper pointed simplicial model category.

Definition 4.3.9. *A weak $\mathcal{G}$-resolution ([15], 6.1) of $A \in \mathcal{C}$ is a $\mathcal{G}$-equivalence $\bar{A} \longrightarrow X^\bullet$ in $c\mathcal{C}$ where $\bar{A}$ is a constant cosimplicial object, such that $X^n \in \mathcal{G}$ for $n \geq 0$.*

If $\mathcal{C}$ is a left proper pointed model category with a class $\mathcal{G}$ of injective models in $Ho(\mathcal{C})$, then Fact 4.3.8 gives the $\mathcal{G}$-resolution model category $c\mathcal{C}^{\mathcal{G}}$.

4.3.2.1 Bousfield-Kan resolution and $\mathcal{R}$-completion

In this section we will describe a construction from ([16], I.2 and [15], 7.2) called the Bousfield-Kan resolution of a space X.

Definition 4.3.10. *Let X be a pointed simplicial set and $\mathcal{R}$ a commutative ring. Let $R : \mathcal{SET}_* \longrightarrow \mathcal{SET}_*$, defined by $(RX)^n$ is the free $\mathcal{R}$-module on X_n modulo the relation $[*] = 0$ (all degenerate simplices are the base point). There are also natural maps*

1. *$\boldsymbol{\phi}_X : X \longrightarrow RX$, given by $x \mapsto 1 \cdot x$, where $\boldsymbol{\phi}_X$ is the inclusion of basis into the simplicial $\mathcal{R}$-module and*
2. *$\boldsymbol{\psi}_X : R^2X \longrightarrow RX$, given by $r_1 \cdot r_2 x \mapsto r_1 r_2 x$, where $\boldsymbol{\psi}_X$ is the multiplication inside the the simplicial $\mathcal{R}$-module.*

Then $(R, \boldsymbol{\phi}, \boldsymbol{\psi})$ is called the Bousfield-Kan triple (monad) ([16], I.2 or [15]) on the category $s\mathcal{SET}_*$.

Note: RX does not inherit an R-module structure if X is not pointed ([16], I 2.2, 2.4).

Using the monad $(R, \boldsymbol{\phi}, \boldsymbol{\psi})$ Bousfield forms a cosimplicial space $R^\bullet X$ and in ([16], I 4.1) it is shown that $X \to R^\bullet X$ is a cosimplicial resolution of X in the sense of Huber [35]. The cosimplicial resolution of X is referred to as the Bousfield-Kan resolution of X.

4.3.2.2 Total objects

Definition 4.3.11. *Let $X \in s\mathcal{C}$, then the n^{th} skeleton ([16], VIII 2.13) of X is a sub-object of X generated by all the simplices of X of dimension $\leq n$. We will denote the n^{th} skeleton of X as $\mathrm{sk}_n X$.*

Definition 4.3.12. *For a cosimplicial space $X^\bullet$, we define the space $\mathrm{Tot}(X^\bullet)$ ([50], page 149) as the space of cosimplicial maps from $\Delta^\bullet \times \Delta[q]$ to $X^\bullet$ (c.f. Notation 2.2.7). That is*

$$\mathrm{Tot}(X^\bullet)_q = Hom_{c\mathcal{CW}_*}(\Delta^\bullet \times \Delta^q, X^\bullet) \tag{4.1}$$

For $s \geq 0$, we define

$$\mathrm{Tot}_s(X^\bullet)_q = Hom_{c\mathcal{CW}_*}(\mathrm{sk}_s\Delta^\bullet \times \Delta^q, X^\bullet). \tag{4.2}$$

Fact 4.3.13. [37] For a cofibration $A^\bullet \longrightarrow Y^\bullet$ and a fibrant $X^\bullet$ the map $Hom(Y^\bullet, X^\bullet) \longrightarrow Hom(A^\bullet, X^\bullet)$ is a fibration.

The inclusion $i_s : \mathrm{sk}_{s-1}\Delta^\bullet \longrightarrow \mathrm{sk}_s\Delta^\bullet$ induces a map $i_s^* : Hom(\mathrm{sk}_s\Delta^\bullet, X^\bullet) \longrightarrow Hom(\mathrm{sk}_{s-1}\Delta^\bullet, X^\bullet)$
By (4.2), we have

$$i_s^* : \mathrm{Tot}_s(X^\bullet) \longrightarrow \mathrm{Tot}_{s-1}(X^\bullet) \tag{4.3}$$

From ([29], VII (4.16)) the inclusion map i_s is a cofibration, then by Fact 4.3.13, it is clear that (4.3) is a fibration. Thus for a cosimplicial space $X^\bullet$, we obtain a tower of fibrations

$$\mathrm{Tot}(X^\bullet) \longrightarrow \cdots \longrightarrow \mathrm{Tot}_k(X^\bullet) \longrightarrow \mathrm{Tot}_{k-1}(X^\bullet) \longrightarrow \cdots \longrightarrow \mathrm{Tot}_0(X^\bullet) \tag{4.4}$$

$\mathrm{Tot}(X^\bullet)$ is the limit of the sequence (4.4),

$$\mathrm{Tot}X^\bullet \cong \varprojlim \mathrm{Tot}_s X^\bullet$$

Definition 4.3.14. *The total space $\mathrm{Tot}(R^\bullet X)$, of the Bousfield-Kan resolution of X, is called the R-completion ([16], I 4.2) of the space X, denoted by $R_\infty X$. Equivalently, $R_\infty X$ is the limit of a tower of fibrations $\{R_s X\}_{s \geq -1}$, where $R_s X = \mathrm{Tot}_s R^\bullet X$.*

Fact 4.3.15. A map $f : X \longrightarrow Y$ induces an isomorphism $H_*(X; \mathcal{R}) \cong H_*(Y; \mathcal{R})$ if and only if f induces a homotopy equivalence $R_\infty X \cong R_\infty Y$ ([16], I 5.5).

Definition 4.3.16. *A space is called R-good ([16], I 5.1) if the map from the space X to its R-completion is a homology isomorphism,*
That is $H_*(X; R) \xrightarrow{\cong} H_*(R_\infty X; R)$.

Example 4.3.17. Let $\mathcal{G}$ contain all the simplicial $\mathcal{R}$-modules. Then the Bousfield-Kan resolution $X \longrightarrow R^{\bullet}X$ in cosimplicial $s\mathcal{SET}_*$ is a weak $\mathcal{G}$-resolution of X in the model category $s\mathcal{SET}_*^{\mathcal{G}}$.

Chapter 5

Free simplicial resolution of $H(\mathcal{R})$-algebras

In this chapter we will give the definition of a free simplicial resolution of $H(\mathcal{R})$-algebra in Definition 5.2.1. A free simplicial resolution of a $H(\mathcal{R})$-algebra Z is a cofibrant simplicial replacement for Z in the model category described in Section 4.1.2. The main result of this chapter is Theorem 5.2.11, and in this theorem we show a simplicial $H(\mathcal{R})$-algebra $Z_\bullet$ augmented by Z constructed in Section 3.5 is a free simplicial resolution for Z.

Definition 5.0.18. *Let $i : A \hookrightarrow X$ be an inclusion. A is a **retract** ([5], 1.4.1) of X if there is a map $r : X \longrightarrow A$ such that $ri = \mathrm{id}_A$.*

Fact 5.0.19. ([35], $\oint 3$) Let $i : A \hookrightarrow X$ be an inclusion and $r : X \longrightarrow A$ be a retract. If $ir \cong id_X$ then $A \cong X$ are homotopy equivalent.

5.1 The natural transformation h

We will prove what Huber refers to as standard method originally devised by Godement [28]. The standard method applies to any category $\mathcal{C}$ if it satisfies the following conditions;

(i) an underlying functor $U : \mathcal{C} \longrightarrow \mathcal{D}$ exists

(ii) a free functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ exists and F is left adjoint to U

(iii) a natural transformation $h : I \longrightarrow FU$ exists such that $\epsilon h = 1$, where ϵ is the counit of adunction.

In Subsection 5.1.1 we will use the free and underlying functors from Section 3.2.1 on $\mathcal{H}(\mathcal{R}) - \mathcal{ALG}$ to show a natural transformation h exists satisfying condition (iii) given above.

5.1.1 h acting on $\mathcal{H}(\mathcal{R})$-algebras

Let Z be an $\mathcal{H}(\mathcal{R})$-algebra, that is a set of graded abelian groups $\{Z^n | n \in \mathbb{N}\}$ with all the primary cohomology operations acting on them. UZ is a graded set since $Z^n = Z(K^n) \in \mathcal{SET}_*$, for each $n \in \mathbb{N}$, so for each $t \in Z^n$,

$$FUZ = [\prod_{n \in \mathbb{N}} \prod_{t \in Z^{n-}} K_t^n, \quad] : \mathcal{H}(\mathcal{R}) \longrightarrow \mathcal{SET}_* .$$

Definition 5.1.1. *We define a natural transformation $h : I \longrightarrow FU$ on Z by $h_Z(t) = pr_t$, for $t \in Z^n$ and $pr_t : \prod_{n \in \mathbb{N}} \prod_{t \in Z^{n-}} K_t^n \longrightarrow K_t^n$.*

In the next lemma we will show that h is a natural transformation.

Lemma 5.1.2. *$h : I \longrightarrow FU$ is a natural transformation on $Z \in \mathcal{H}(\mathcal{R}) - \mathcal{ALG}$.*

Proof.
Let $t \in X$ and $w : X \longrightarrow Z$. So we need to show the square 5.1 is commutative.

$$\begin{tikzcd} X \arrow[r, "h_X"] \arrow[d, "w"] & FU(X) \arrow[d, "FU(w)"] \\ Z \arrow[r, "h_Z"] & FU(Z) \end{tikzcd} \tag{5.1}$$

By definition we have $h_X(t) = pr_t$ and $h_Z(w(t)) = pr_{w(t)}$ which are cohomology classes. From Section 3.2.1 we know $FU(w)$ is induced by a product map sending $K^n_{w(t)}$ to $\prod_{w(t_i)=w(t)} K^n_{t_i}$ which in particular maps $K^n_{w(t)}$ isomorphically to K^n_t. Hence $FU(w)$ takes pr_t to $pr_{w(t)}$.

Therefore, the square commutes. $\square$

Lemma 5.1.3. *Let $\epsilon : FU \longrightarrow I$ be the counit of adjunction for any $Z \in \mathcal{H}(\mathcal{R}) - \mathcal{ALG}$, given in Section 3.2.3, then $\epsilon_Z h_Z = 1_Z$.*

Proof. To show $\epsilon_Z h_Z = 1_Z$, by definition of h_Z and ϵ_Z we have for any $Z \in \mathcal{H}(\mathcal{R}) - \mathcal{ALG}$

$$
\begin{aligned}
\epsilon_Z(h_Z(t)) &= \epsilon_Z(pr_t) \\
&= t \\
\text{so } \epsilon_Z h_Z &= 1_Z.
\end{aligned}
$$

$\square$

5.1.2 Induced maps

From Section 3.5 we know the face and degeneracy maps of a simplicial $H(\mathcal{R})$-algebra induces a face and degeneracy maps of simplicial abelian groups (c.f. Figure 3.3). We will denote the face and degeneracy maps of vertical strip (j^{th} simplicial abelian group) in Figure 3.3 by

(i) $\epsilon_{Z^j} : FU(Z^j) \longrightarrow Z^j$ is the augmentation.

(ii) ${d^n_i}_{Z^j} : FU^{n+1}(Z^j) \longrightarrow FU^n(Z^j)$ is the i^{th} face map

(iii) ${s^n_i}_{Z^j} : FU^{n+1}(Z^j) \longrightarrow FU^{n+2}(Z^j)$ is the i^{th} degeneracy map.

As h in Definition 5.1.1 is a natural transformation we have an induced map on the j^{th} simplicial abelian group which will be denoted by $h_{Z^j} : Z^j \longrightarrow FU(Z^j)$. Lemma 5.1.2 and Lemma 5.1.3 implies h_{Z^j} satisfies the condition (iii). That is $\epsilon_{Z^j} h_{Z^j} = 1_{Z^j}$. Diagram (5.2) shows the j^{th} simplicial abelian group of Figure 3.3.

$$\begin{tikzcd} Z^j & FU(Z^j) \arrow[l, "\epsilon_{Z^j}"'] \arrow[rr, "s^0_{0Z^j}"] & & FU^2(Z^j) \arrow[ll, shift right, "d^1_{1Z^j}"'] \arrow[ll, shift left, "d^1_{0Z^j}"] \arrow[r, shift left, "s^1_{1Z^j}"] \arrow[r, shift right, "s^1_{0Z^j}"'] & FU^3(Z^j) \arrow[l, shift right=2, "d^2_{2Z^j}"'] \arrow[l, "d^2_{1Z^j}" description] \arrow[l, shift left=2, "d^2_{0Z^j}"] \cdots \end{tikzcd} \tag{5.2}$$

5.2 Construction of a free-underlying resolution

Dual to Stover ([53], 5.4), we will define a free simplicial resolution of a $H(\mathcal{R})$-algebra.

Definition 5.2.1. *A free simplicial resolution of a $H(\mathcal{R})$-algebra Z is a simplicial $H(\mathcal{R})$-algebra $F_\bullet$ augmented by Z with the following properties:*

(i) $\pi_p F_\bullet = 0$ *for* $p > 0$

(ii) $\pi_p F_\bullet = Z$ *for* $p = 0$

(iii) F_n *is a free $H(\mathcal{R})$-algebra for each* $n \geq 0$.

Our Theorem 5.2.11 is an analogous result to Huber's ([35], Theorem 3.2) or Weibel's ([54], Proposition 8.6.10). In Huber's proof to show the simplicial object $(FU)_\bullet(Z) \in \mathcal{C}$ is a free resolution he uses a functor $T : \mathcal{C} \longrightarrow \mathcal{SET}_*$, which satisfies a condition he calls T-triviality ([35], Definition 3.1) and then he shows the simplicial set $T((FU)_\bullet(Z))$ is weakly homotopy equivalent to the constant simplicial set of Z. Huber defines a map f to be a weak equivalence when the underlying map in pointed simplicial sets is weakly homotopy equivalent as a Kan complex. Finally he explains how to put a group structure on these underlying simplicial sets.

At the time Huber wrote his paper [35], model category theory [47] and Lawvere's theory of algebraic structures [39] were not known. Huber's result from the point of view of model category theory, can be applied to construct resolutions in the model category structure described in Fact 4.1.5 (which is a generalization of the model category given by Quillen in ([47] II.4 Theorem 4)). Resolutions constructed using our Theorem 5.2.11 can be applied if we are working in the model category structure described in Section 4.1.2. This is the main difference between our Theorem 5.2.11 and Huber's result, because the notion of a weak equivalence is different for the two model categories.

Weibel's Proposition 8.6.10, says that, given a pair of adjoint functors with F left adjoint to U and $U : \mathcal{C} \longrightarrow \mathcal{A}b$, then for every $Z \in \mathcal{C}$, the underlying set $U(FU)_{\bullet}Z) \longrightarrow U(Z)$ is acyclic. Just like Huber's result this can be applied to Fact 4.1.5 but not to the results of Section 4.1.2. Also one other difference between our Theorem 5.2.11 and Weibel's result is that to apply Weibel's result, the target category for the underlying functor needs to be $\mathcal{A}b$ and in our case we need a pair of adjoint functors with the target of underlying functor to any category that has a underlying set structure.

The first row of Figure 5.1 are functors and natural transformations giving simplicial abelian groups as in diagram (5.2) when applied to Z^j from Figure 3.3. The second row of Figure 5.1 are functors and natural transformations giving the constant simplicial group [1] when applied to an Z^j.

[1] Constant simplicial obect [54], 8.1.1

$$
\begin{tikzcd}[column sep=large]
\cdots FU^3 \arrow[r, shift left=2, "d_0^2"] \arrow[r, "d_1^2"] \arrow[r, shift right=2, "d_2^2"'] \arrow[d, "d_0^0 d_0^1 d_0^2"'] & FU^2 \arrow[l, shift left=4, "s_0^1"'] \arrow[l, shift right=4, "s_1^1"] \arrow[r, shift left=1, "d_0^1"] \arrow[r, shift right=1, "d_1^1"'] \arrow[d, "d_0^0 d_0^1"] & FU \arrow[l, shift left=3, "s_0^0"'] \arrow[r, "d_0^0"] \arrow[d, "d_0^0"] & I \arrow[d, "id"] & \text{———①} \\
\cdots I \arrow[r, leftrightarrow, "id"] \arrow[d, "s_0^1 s_0^0 h"'] & I \arrow[r, leftrightarrow, "id"] \arrow[d, "s_0^0 h"] & I \arrow[r, leftrightarrow, "id"] \arrow[d, "h"] & I \arrow[d, "id"] & \text{———②} \\
\cdots FU^3 \arrow[r, shift left=2, "d_0^2"] \arrow[r, "d_1^2"] \arrow[r, shift right=2, "d_2^2"'] \arrow[d, "d_0^0 d_0^1 d_0^2"'] & FU^2 \arrow[l, shift left=4, "s_0^1"'] \arrow[l, shift right=4, "s_1^1"] \arrow[r, shift left=1, "d_0^1"] \arrow[r, shift right=1, "d_1^1"'] \arrow[d, "d_0^0 d_0^1"] & FU \arrow[l, shift left=3, "s_0^0"'] \arrow[r, "d_0^0"] \arrow[d, "d_0^0"] & I \arrow[d, "id"] & \text{———③} \\
\cdots I \arrow[r, leftrightarrow, "id"] & I \arrow[r, leftrightarrow, "id"] & I \arrow[r, leftrightarrow, "id"] & I & \text{———④}
\end{tikzcd}
$$

Figure 5.1

Lemma 5.2.2. *Figure 5.1 is a commutative diagram.*

Proof. A square of the form

$$
\begin{tikzcd}[column sep=huge]
FU^{n+1} \arrow[r, shift left=3, "d_0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_n"'] \arrow[d, "(d_0)^{n+1}"'] & FU^n \arrow[d, "(d_0)^n"] \\
I \arrow[r, leftrightarrow, "id"] & I
\end{tikzcd}
$$

is commutative because

$$
\begin{aligned}
(d_0)^n d_k &= (d_0)^{n-k}(d_0)^{k-1} d_0 d_k \\
&= (d_0)^{n-k}(d_0)^{k-1} d_{k-1} d_0 \\
&= (d_0)^{n-k}(d_0)^{k-2} d_0 d_{k-1} d_0 \qquad (\text{using } \; d_i^n d_j^{n+1} = d_{j-1}^n d_i^{n+1} \; \text{ for } \; i < j) \\
&= (d_0)^{n-k}(d_0)^{k-2} d_{k-2} (d_0)^2 \\
&\vdots \quad (\text{iterating } (k-2) \text{ times}) \\
&= (d_0)^{n-k} d_0 d_0 (d_0)^{k-1} \\
&= (d_0)^{n+1} \\
&= (id)(d_0)^{n+1}, \;\; \text{where } \; 0 \le k \le n.
\end{aligned}
$$

Similarly

$$\begin{tikzcd}
FU^{n+1} \arrow[d, "(d_0)^{n+1}"'] & & FU^n \arrow[ll, shift right=3, "s_0"'] \arrow[ll, phantom, "\vdots"] \arrow[ll, shift left=3, "s_{n-1}"] \arrow[d, "(d_0)^n"] \\
I \arrow[rr, leftrightarrow, "id"] & & I
\end{tikzcd}$$

is commutative because

$$\begin{aligned}
(d_0)^{n+1}s_k &= (d_0)^{n-(k+1)}(d_0)^k d_0 s_k \\
&= (d_0)^{n-(k+1)}(d_0)^k s_{k-1} d_0 \\
&= (d_0)^{n-(k+1)}(d_0)^{k-1} d_0 s_{k-1} d_0 \qquad (\text{using } \ d_i^{n+1}s_j^n = s_{j-1}^{n-1}d_i^n \ \text{ for } \ i<j) \\
&= (d_0)^{n-(k+1)}(d_0)^{k-1} s_{k-2} (d_0)^2 \\
&\vdots \ \ (\text{iterating } (k-2) \text{ times}) \\
&= (d_0)^{n-(k+1)}(d_0 s_0)(d_0)^k \qquad (\text{using } \ d_i^{n+1}s_i^n = id) \\
&= (d_0)^n \\
&= (id)(d_0)^n, \ \ \text{where } \ 0 \le k \le n-1.
\end{aligned}$$

Also a square of the form

$$\begin{tikzcd}
I \arrow[rr, leftrightarrow, "id"] \arrow[d, "(s_0)^n h"'] & & I \arrow[d, "(s_0)^{n-1}h"] \\
FU^{n+1} \arrow[rr, shift left=3, "d_0"] \arrow[rr, phantom, "\vdots"] \arrow[rr, shift right=3, "d_n"'] & & FU^n
\end{tikzcd}$$

is commutative because

$$\begin{aligned}
d_k(s_0)^n h &= (d_k s_0)(s_0)^{n-1}h \\
&= (s_0 d_{k-1})(s_0)^{n-1}h \qquad (\text{using } \ d_i^{n+1}s_j^n = s_j^{n-1}d_{i-1}^n \ \text{ for } \ i>j+1) \\
&\vdots \ \ (\text{iterating } (k-1) \text{ times}) \\
&= (s_0)^{k-1}(d_1 s_0)(s_0)^{n-k}h \qquad (\text{using } \ d_{i+1}^{n+1}s_i^n = id) \\
&= (s_0)^{n-1}h \\
&= (s_0)^{n-1}h(id), \ \ \text{where } \ 0 \le k \le n-1.
\end{aligned}$$

Finally a square of the form

$$\begin{tikzcd}
I \arrow[rr, leftrightarrow, "id"] \arrow[d, "(s_0)^{n+1}h"'] & & I \arrow[d, "(s_0)^n h"] \\
FU^{n+2} & \vdots & FU^{n+1} \arrow[ll, shift right=4, "s_0"'] \arrow[ll, shift left=4, "s_n"]
\end{tikzcd}$$

is commutative because

$$\begin{aligned}
s_k(s_0)^n h &= (s_k s_0)(s_0)^{n-1}h \\
&= (s_0 s_{k-1})(s_0)^{n-1}h \qquad (\text{using } \ s_i^{n+1}s_j^n = s_{j+1}^{n+1}s_i^n \ \text{ for } \ i \leq j) \\
&= (s_0)^2 s_{k-2}(s_0)^{n-2}h \\
&\vdots \ \ (\text{iterating } (k-2) \text{ times}) \\
&= (s_0)^k s_0 (s_0)^{n-k}h \\
&= (s_0)^{n+1}h \\
&= (s_0)^{n+1}h(id), \ \ \text{where } \ 0 \leq k \leq n.
\end{aligned}$$

This concludes the diagram given above being commutative. □

Example 5.2.3. If we apply the simplicial functor ① in Lemma 5.2.2 to Z^j we get the simplicial abelian group (5.2). Then, Lemma 5.2.2 shows ①⟶②, ②⟶③ and ③⟶③ is a map of simplicial abelian groups (2.2.8).

Let f be the map ①⟶②⟶③, that is, $FU^{n+1} \xrightarrow{(d_0)^{n+1}} I \xrightarrow{(s_0)^n h} FU^{n+1}$ in Figure 5.1 and f is defined explicitly by

$$\begin{aligned}
f_{-1} &= 1 \\
f_0 &= hd_0^0 \\
f_n &= (s_0)^n h (d_0)^{n+1}.
\end{aligned}$$

Lemma 5.2.4. *The chain map ②⟶③⟶④ is the identity* 1.

Proof.

$$
\begin{aligned}
(d_0)^{n+1}(s_0)^n h &= d_0 \underbrace{(d_0 \ldots (d_0(d_0}_{n\,\text{times}} \overbrace{s_0)s_0) \ldots s_0)}^{n\,\text{times}} h \\
&= d_0 h \\
&= \epsilon h \\
&= 1 \quad \text{by Lemma 5.1.3.}
\end{aligned}
$$

□

Remark 5.2.5. *If the chain map ②⟶③ of Lemma 5.2.4 is an inclusion then according to Definition 5.0.18, ② is a retract of ③.*

Fact 5.2.6. ([41] I.5) Let f and g be morphisms in a category where $fg = id$ then f is epic and g is monic. In the category of abelian groups f epic is the same as f is surjective and g is monic is the same as g is injective.

Lemma 5.2.7 is only applicable to categories where inclusion map is defined.

Lemma 5.2.7. *The chain maps ②⟶③ at Z^j (c.f. Example 5.2.3) is an inclusion map.*

Proof. From Lemma 5.2.4 and Fact 5.2.6 it is clear that ②⟶③ is monic, hence a 1-1 map in the category of abelian groups. □

The next step is to show that f is homotopic to identity chain map $id_{①}$ from ①⟶③. A homotopy between f and $id_{①}$ can be given by the sequence of maps $h_i^n : FU^{n+1}(Z) \longrightarrow FU^{n+2}(Z)$, for $n \geq 0$, defined by

$$
\begin{aligned}
h_0^{-1} &= h \\
h_0^n &= hFU^{n+1} : IFU^{n+1} \longrightarrow FUFU^{n+1} \\
h_i^n &= (s_0)^i h_0^{n-i} (d_0)^i \ \text{ where } \ 0 < i \leq n
\end{aligned}
$$

First we will prove Lemma 5.2.8 which will be used in Lemma 5.2.9.

Lemma 5.2.8. *(I)* $d^{n+1}_{i+1}h^n_0 = h^{n-1}_0 d^n_i$

(II) $s^{n+1}_{i+1}h^n_0 = h^{n+1}_0 s^n_i$

Proof. (I)

$$\begin{aligned}
d^{n+1}_{i+1}h^n_0 &= FU^{i+1}\epsilon FU^{(n+1)-(i+1)}hFU^{n+1} \\
&= FUFU^i\epsilon FU^{n-i}hFU^iFUFU^{n-i} \\
&= (FUFU^i\epsilon hFU^iFU)FU^{n-i} \quad \text{by equation (2.4)} \\
&= (hFU^iFU^i\epsilon)FU^{n-i} \quad \text{by Example 2.1.4} \qquad (5.3) \\
&= hFU^iFU^{n-i}FU^i\epsilon FU^{n-i} \quad \text{by equation (2.4)} \\
&= hFU^nFU^i\epsilon FU^{n-i} \\
&= h^{n-1}_0 d^n_i
\end{aligned}$$

(II)

$$\begin{aligned}
s^{n+1}_{i+1}h^n_0 &= FU^{i+1}\nu FU^{n-i}hFU^{n+1} \\
&= FUFU^i\nu FU^{n-i}hFU^iFUFU^{n-i} \\
&= (FUFU^i\nu hFU^iFU)FU^{n-i} \quad \text{by equation (2.4)} \\
&= (hFU^iFU^2FU^i\nu)FU^{n-i} \quad \text{by Example 2.1.5} \qquad (5.4) \\
&= hFU^iFU^2FU^{n-i}FU^i\nu FU^{n-i} \quad \text{by equation (2.4)} \\
&= hFU^{n+2}FU^i\nu FU^{n-i} \\
&= h^{n+1}_0 s^n_i
\end{aligned}$$

□

Lemma 5.2.9. *Let f be the map ①⟶②⟶③, that is, $FU \longrightarrow I \longrightarrow FU$ in Figure 5.1 and $id_{①}$ be identity chain map $FU^{n+1} \longrightarrow FU^{n+1}$.*
Then f is simplically homotopic $id_{①}$.

Proof.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdots FU^3 \arrow[r, shift left=4, "d_0^2"] \arrow[r, shift left=0, "d_1^2"] \arrow[r, shift right=4, "d_2^2"'] & FU^2 \arrow[l, shift left=2, "s_0^1"] \arrow[l, shift right=2, "s_0^1"'] \arrow[r, shift left=2, "d_0^1"] \arrow[r, shift right=2, "d_1^1"'] \arrow[dl, "{h_1^1, h_0^1}"] & FU \arrow[l, "s_0^0"] \arrow[r, "d_0^0"] \arrow[dl, "h_0^0"] & I \arrow[dl, "h_0^{-1}"] \\
\cdots FU^3 \arrow[r, shift left=4, "d_0^2"] \arrow[r, shift left=0, "d_1^2"] \arrow[r, shift right=4, "d_2^2"'] & FU^2 \arrow[l, shift left=2, "s_0^1"] \arrow[l, shift right=2, "s_0^1"'] \arrow[r, shift left=2, "d_0^1"] \arrow[r, shift right=2, "d_1^1"'] & FU \arrow[l, "s_0^0"] \arrow[r, "d_0^0"] & I
\end{tikzcd}
$$

(A)

$$
\begin{aligned}
d_0^0 h_0^{-1} &= \epsilon h = 1 \text{ where } 1 : I \longrightarrow I \text{ is the identity natural transformation} \\
d_0^{n+1} h_0^n &= \epsilon F U^{n+1} h F U^{n+1} \\
&= \epsilon h F U^{n+1} \\
&= 1 F U^{n+1}
\end{aligned}
$$

where $1FU^{n+1} : FU^{n+1} \longrightarrow FU^{n+1}$ is the identity natural transformation. So all of these maps gives $id_{①}$.

(B)

$$
\begin{aligned}
d_{n+1}^{n+1} h_n^n &= d_{n+1}^{n+1} (s_0)^n h_0^0 (d_0)^n \\
&= d_{n+1}^{n+1} s_0 (s_0)^{n-1} h_0^0 (d_0)^n \\
&= s_0 d_n (s_0)^{n-1} h_0^0 (d_0)^n \\
&\vdots \ \ (\text{iterating } d_n s_0 = s_0 d_{n-1} \ \ (n-1)\,\text{times}) \\
&= (s_0)^n d_1 h_0^0 (d_0)^n \\
&= (s_0)^n h_0^{-1} d_0 (d_0)^n \qquad \text{using } (5.3) \\
&= (s_0)^n h (d_0)^{n+1} \\
&= f_n
\end{aligned}
$$

(C)

$$
\begin{aligned}
d_i h_j^n &= d_i(s_0)^j h_{n-j}^0 (d_0)^j \\
&= d_i(s_0)^i (s_0)^{j-i} h_{n-j}^0 (d_0)^{j-(i+1)} d_0 (d_0)^i \\
&= s_0 d_{i-1}(s_0)^{i-1} (s_0)^{j-i} h_{n-j}^0 (d_0)^{j-(i+1)} d_0 d_1 (d_0)^{i-1} \\
&\vdots \quad \text{(iterating } (i-2) \text{ times)} \\
&= (s_0)^{i-1} d_1 s_0 (s_0)^{j-i} h_{n-j}^0 (d_0)^{j-(i+1)} (d_0)^{i-1} d_{i-1}(d_0) \\
&= ((s_0)^{i-1}(s_0)^{j-i}) h_{n-j}^0 ((d_0)^{j-(i+1)} (d_0)^{i-1} d_0) d_i \\
&= ((s_0)^{j-1}) h_{(n-1)-(j-1)}^0 ((d_0)^{j-1}) d_i \qquad \text{where } n-j = (n-1)-(j-1) \\
&= h_{j-1}^{n-1} d_i
\end{aligned}
$$

(D)

$$
\begin{aligned}
d_j h_j^n &= d_j (s_0)^j h_0^{n-j} (d_0)^j \\
&= d_j s_0 (s_0)^{j-1} h_0^{n-j} d_0 (d_0)^{j-1} \\
&= s_0 d_{j-1} (s_0)^{j-1} h_0^{n-j} d_0 (d_0)^{j-1} \\
&\vdots \quad \text{(iterating } (j-2) \text{ times)} \\
&= (s_0)^{j-1} d_1 (s_0) h_0^{n-j} d_0 (d_0)^{j-1} \\
&= (s_0)^{j-1} h_0^{n-j} d_0 (d_0)^{j-1} \qquad \text{using } d_1(s_0) = id \\
&= (s_0)^{j-2} s_0 d_1 h_0^{n-j+1} (d_0)^{j-1} \qquad \text{by } d_1(s_0) = id \text{ and } (5.3) \\
&= (s_0)^{j-2} d_2 s_0 h_0^{n-j+1} (d_0)^{j-1} \qquad \text{using } s_0 d_1 = d_2 s_0 \\
&\vdots \quad \text{(iterating } (j-2) \text{ times)} \\
&= d_j (s_0)^{j-1} h_0^{n-(j-1)} (d_0)^{j-1} \\
&= d_j h_{j-1}^n
\end{aligned}
$$

(E)

$$
\begin{aligned}
d_i h_j^n &= d_i (s_0)^j h_0^{n-j} (d_0)^j \quad \text{where} \quad i > j+1 \\
&= d_i s_0 (s_0)^{j-1} h_0^{n-j} (d_0)^j \\
&= s_0 d_{i-1} (s_0)^{j-1} h_0^{n-j} (d_0)^j \\
&\vdots \quad \text{(iterating } (j-1) \text{ times)} \\
&= (s_0)^j d_{i-j} h_0^{n-j} (d_0)^j \\
&= (s_0)^j h_0^{n-1-j} d_{i-1-j} d_0 (d_0)^{j-1} \qquad\qquad \text{by (5.3)} \\
&= (s_0)^j h_0^{n-1-j} d_0 d_{i-1-j+1} (d_0)^{j-1} \\
&\vdots \quad \text{(iterating } (j-1) \text{ times)} \\
&= (s_0)^j h_0^{n-1-j} (d_0)^j d_{i-1-j+j} \\
&= h_j^{n-1} d_{i-1}
\end{aligned}
$$

(F)

$$
\begin{aligned}
s_i h_j^n &= s_i (s_0)^j h_0^{n-j} (d_0)^j \quad \text{where} \quad (i \leq j) \\
&= s_i s_0 (s_0)^{j-1} h_0^{n-j} (d_0)^{j-i} (d_0 s_0) d_0 (d_0)^{i-1} \qquad \text{using} \quad d_0(s_0) = id \\
&= s_0 s_{i-1} (s_0)^{j-1} h_0^{n-j} (d_0)^{j-i} (d_0)^2 s_1 (d_0)^{i-1} \\
&\vdots \quad \text{(iterating } (i-1) \text{ times)} \\
&= (s_0)^i s_0 (s_0)^{j-1} h_0^{n-j} (d_0)^{j-i} (d_0)^{i+1} s_i \\
&= (s_0)^{j+1} h_0^{n+1-(j+1)} (d_0)^{j+1} s_i \\
&= h_{j+1}^{n+1} s_i
\end{aligned}
$$

(G)

$$
\begin{aligned}
h_j^{n+1} s_{i-1} &= (s_0)^j h_0^{(n+1)-j} (d_0)^j s_{i-1} \text{ where } (i<j) \\
&= (s_0)^j h_0^{(n+1)-j} (d_0)^{j-1} d_0 s_{i-1} \\
&= (s_0)^j h_0^{(n+1)-j} (d_0)^{j-1} s_{i-1-1} d_0 \\
&\vdots \ \text{(iterating } (j-1) \text{ times)} \\
&= (s_0)^{j-1} s_0 h_0^{(n+1)-j} s_{i-1-j} (d_0)^j \\
&= (s_0)^{j-1} s_0 s_{i-j} h_0^{n-j} (d_0)^j \qquad \text{by (5.4)} \\
&= (s_0)^{j-1} s_{i-j+1} s_0 h_0^{n-j} (d_0)^j \\
&\vdots \ \text{(iterating } (j-1) \text{ times)} \\
&= s_{i-j+j} (s_0)^j h_0^{n-j} (d_0)^j \\
&= s_i (s_0)^j h_0^{n-j} (d_0)^j \\
&= s_i h_j^n
\end{aligned}
$$

□

Remark 5.2.10. *From Lemma 5.2.7, the map ②⟶ ③ is an inclusion and by Lemma 5.2.4, ② is a retract of ① then by Fact 5.0.19, ① and ② have the same homotopy type (that is the corresponding homotopy groups are the same) when the maps f and $id_{①}$ are homotopic.*

Theorem 5.2.11. *Let Z be a $H(\mathcal{R})$-algebra and $(FU)_\bullet(Z)$ be the free simplicial $H(\mathcal{R})$-algebra augmented by Z, constructed using the adjoint pair $\mathcal{SET}_* \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{H}(\mathcal{R}) - \mathcal{ALG}$. Then $(FU)_\bullet(Z)$ is a free simplicial resolution of Z in the sense of Definition 5.2.1.*

Proof. From Section 3.5 an augmented simplicial $H(\mathcal{R})$-algebra $(FU)_\bullet(Z)$ can be thought of as simplicial abelian groups in $s\mathcal{SET}$. From Remark 5.2.10 the j^{th} simplicial abelian group from diagram (5.2) is homotopy equivalent to the constant simlicial abelian group. Instead of calculating the homotopy groups of simplicial abelian group (5.2) we will calculate the homotopy groups of the constant simplicial abelian group, since (5.2) has the same homotopy type as the constant simplicial abelian group by Lemma 5.2.9 and Remark 5.2.10.

First, we form the Moore chain complex for the augmented constant simplicial abelian group

$$Z^j \xleftrightarrow{\ id\ } Z^j \xleftrightarrow{\ id\ } Z^j \xleftrightarrow{\ id\ } Z^j \cdots \tag{5.5}$$

as explained in (2.33). Then (5.5) simplifies to

$$0 \longleftarrow Z^j \xleftarrow{id} Z^j \xleftarrow{\ \ 0\ \ } Z^j \xleftarrow{\ \ id\ \ } Z^j \cdots \tag{5.6}$$

since all d_i's are identity maps.

The homology groups of the chain complex (5.6) is either

(i) $ker(0)/im(id) = Z^j/Z^j \cong 0$

(ii) $ker(id)/im(0) = 0/0 \cong 0$

This means the chain complex (5.6) is exact everywhere and by Definition 2.2.17 and Fact 2.2.18 the zeroth homology is the augmentation. Also as h is a homotopy the two augmented Moore chain complexes (5.6) and the Moore chain complex corresponding to (5.2) have the same homology. Therefore the Moore chain complex corresponding to (5.2) is exact.
Hence by Fact 2.2.18

$$\pi_i(FU)_\bullet(Z^j) = 0 \quad \text{if} \ \ i > 0$$

$$\pi_0(FU)_\bullet(Z^j) = Z^j$$

The composition of the simplicial functor $FU_\bullet Z : \mathcal{H}(\mathcal{R}) \longrightarrow \mathcal{A}b$ followed by the homotopy $\pi_0 : \mathcal{A}b \longrightarrow \mathcal{SET}_*$ defines a functor $\mathcal{H}(\mathcal{R}) \longrightarrow \mathcal{SET}_*$, therefore defining an $H(\mathcal{R})$-algebra.

From Figure 3.3 where each row is an $\mathcal{H}(\mathcal{R})$-algebra we have $d_{aug_Z} = d_0^0 : FUZ \longrightarrow Z$ is a morphism of $\mathcal{H}(\mathcal{R})$-algebras.

$$\begin{tikzcd} \mathcal{H}(\mathcal{R}) \arrow[rr, bend left, "FUZ"] \arrow[rr, bend right, "Z"'] \arrow[rr, phantom, "\Downarrow d_{aug_Z}"] & & \mathcal{SET} \end{tikzcd} \tag{5.7}$$

Now we have the isomorphism $\pi_0 FUZ \xrightarrow{\cong} Z$ (on each column of Figure 3.3) induced by d_{aug}, a morphism of $\mathcal{H}(\mathcal{R})$-algebras. By naturality of cohomology operations $\pi_0 FUZ$ has the same $H(\mathcal{R})$-algebra structure as Z. $\square$

Chapter 6

A cosimplicial resolution of a space

The purpose of this Chapter is to show how to construct a cosimplicial resolution of a space X using products of Eilenberg-Mac Lane spaces.
First we show a functorial construction T on $\mathcal{CW}_*$ and we show the construction $T(X)$ is homotopy equivalent to a product of Eilenberg-Mac Lane spaces. There are two natural maps ς and β which can be constructed together with T and the triple (T, ς, β) forms a monad on spaces. After that we apply Huber's standard construction [35] on the monad (T, ς, β) to form a cosimplicial space. Then, in Theorem 6.3.2 we show this cosimplicial space is a resolution in the sense of Huber [35]. Finally in Theorem 6.3.5 we show cohomology $\mathcal{H}(\mathcal{R})$-algebra of this cosimplicial resolution is augmented by the $\mathcal{H}(\mathcal{R})$-algebra of X.

6.1 Construction $T(X)$

It is known from elementary homotopy theory ([5], Proposition 1.4.9) that every null-homotopic map $f_i^X : X \longrightarrow K^q$, indexed by a set $i \in S$, factors through a path space PK^q, but for each $i \in S$, f_i^X may factor through PK^q indexed by a map $g_{i_j}^X$ where $j \in J$ indexes the number of null-homotopies for each $i \in S$. The map $ev : PK^q \longrightarrow K^q$ in diagram (6.1) is the evaluation map.

$$\begin{tikzcd} & PK^q \arrow[dr, "ev"] & \\ X \arrow[ur, "g_{i_j}^X"] \arrow[rr, "f_i^X"'] & & K^q \end{tikzcd} \tag{6.1}$$

Conversely, given any map $g_{i_j}^X : X \longrightarrow PK^q$ the composition $ev\, g_{i_j}^X$ is null-homotopic.

6.1.1 The maps ϕ and e

We let $\phi : \prod_{q\in\mathbb{N}} \prod_{f_i^X : X\to K^q} K^q_{f_i^X} \longrightarrow \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X : X\to PK^q} K^q_{g_{i_j}^X}$ be the map described as follows.

For a map $f : X \longrightarrow K^q$, if it is null-homotopic it has a factorization through the path space as shown in diagram (6.1). But, for every null-homotopic map there may be many null-homotopies. So, for each null-homotopic $f_i^X : X \longrightarrow K^q$ we can construct a map $\phi_{f_i^X}$

$$\begin{tikzcd} & K^q_{f_i^X} & \\ K^q_{f_i^X} \arrow[ur, "id"] \arrow[r, "\{id\}_{f_i^X}"'] \arrow[rr, bend right, "\phi_{f_i^X}"'] & \prod_{f_i^X = ev\, g_{i_j}^X : X\to K^q} K^q_{ev\, g_{i_j}^X} \arrow[u, "\text{project}"'] \arrow[r, "\text{identify}"] & \prod_{g_{i_j}^X : X\to PK^q} K^q_{g_{i_j}^X} \end{tikzcd} \tag{6.2}$$

Then ϕ factors through $\prod_{q\in\mathbb{N}} \prod_{f_i^X \simeq *} K^q_{f_i^X}$ followed by $\prod_{q\in\mathbb{N}} \prod_{f_i^X \simeq *} \phi_{f_i^X}$.

Let e be the map $\prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}} \longrightarrow \prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} K^q_{g^X_{i_j}}$ such that e evaluates each path in $\prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}}$ at its end point.

6.1.2 $T(X)$

For any space $X \in \mathcal{CW}_*$, we define $T(X)$ as the pullback of ϕ and e as shown in diagram (6.3).

$$\begin{tikzcd}
T(X) \arrow[r, "p_{2X}\, inc"] \arrow[d, "p_{1X}\, inc"'] & \prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}} \arrow[d, "e"] \\
\prod_{q\in\mathbb{N}}\prod_{f^X_i:X\to K^q} K^q_{f^X_i} \arrow[r, "\phi"] & \prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} K^q_{g^X_{i_j}}
\end{tikzcd} \tag{6.3}$$

Analogous to the explanation given in Section 2.1.6, we will think of the pullback $T(X)$ as an equalizer to the diagram

$$\prod_{q\in\mathbb{N}}\prod_{f^X_i:X\to K^q} K^q_{f^X_i} \times \prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}} \underset{\phi\, p_{1X}}{\overset{e\, p_{2X}}{\rightrightarrows}} \prod_{q\in\mathbb{N}}\prod_{g:X\to PK^q} K^q_{g^X_{i_j}}\,. \tag{6.4}$$

In diagram (6.3), the map inc is the inclusion of $T(X)$ into the product

$$\prod_{q\in\mathbb{N}}\prod_{f^X_i:X\to K^q} K^q_{f^X_i} \times \prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}}. \tag{6.5}$$

The maps p_{1X} and p_{2X} are the projections from (6.5) onto $\prod_{q\in\mathbb{N}}\prod_{f^X_i:X\to K^q} K^q_{f^X_i}$ and $\prod_{q\in\mathbb{N}}\prod_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}}$ respectively. We will write $P_{1X} := p_{1X}\, inc$ and $P_{2X} := p_{2X}\, inc$.

6.1.3 Geometric description of $T(X)$

$T(X)$ is the equalizer of the diagram (6.4), so from ([6], pg 105) we know that $T(X)$ is a sub-object of the product given in (6.5) . Since $T(X)$ is a sub-object of the product (6.5) we can think of $T(X)$ as a subspace of the product (6.5). This subspace consists of factors of K^q_f indexed by non null-homotopic maps and also those indexed by null-homotopic maps. The points in the connected component of each factor of K^q_f indexed by a null-homotopic map $f = ev\ g_j\,,\ \ j \in J$ (by (6.1)) are identified with the end points of the paths of all the path spaces $PK^q_{g_j}$. Note that a single null-homotopic map $f : X \longrightarrow K^q$ can have many maps $g_j : X \longrightarrow PK^q$ factoring through the same path space.

Proposition 6.1.1. *$T(X)$ is homotopy equivalent to a product Eilenberg-Mac Lane spaces*

Proof. From (6.5) and the description given above we know the space $T(X)$ is a subspace of Eilenberg-Mac Lane spaces K^q_f and path spaces PK^q_g. Contracting any path space leaves the set of loops on K^q, but since $\Omega K^q \cong K^{q-1}$, we have $T(X)$ is homotopy equivalent to a product of Eilenberg-Mac Lane spaces. □

Remark 6.1.2. *Given that $T(X)$ is a sub-object of (6.5), in the proofs to follow we will use factors of $K^q_f \subset T(X)$ and $PK^q_g \subset T(X)$ even though for null homotopic maps $f = ev\ g$ the two spaces K^q_f and PK^q_g are identified as described above.*

6.1.4 T acting on maps

Before we define $T(X)$ as a functor we need to define $T(X)$ on maps. For a map $b : X \longrightarrow Y$ in $\mathcal{CW}_*$, we define $T(b) : T(X) \longrightarrow T(Y)$ as described below.

First we construct $T(Y)$ as in diagram (6.3). If $f^Y_i : Y \longrightarrow K^q$ and $g^Y_{i_j} : Y \longrightarrow$

PK^q then by composing b with these maps we get the maps $f_i^Y b : X \longrightarrow Y \longrightarrow K^q$ and $g_{i_j}^Y b : X \longrightarrow Y \longrightarrow PK^q$.

Since

$$\prod_{q\in\mathbb{N}} \prod_{f_i^Y b:X\to K^q} K^q_{f_i^Y b} \subset \prod_{q\in\mathbb{N}} \prod_{f_i^X:X\to K^q} K^q_{f_i^X}$$

and

$$\prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y b:X\to PK^q} PK^q_{g_{i_j}Yb} \subset \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X:X\to PK^q} PK^q_{g_{i_j}^X} \; ,$$

we have the maps

$$T(X) \xrightarrow{P_{1X}} \prod_{q\in\mathbb{N}} \prod_{f_i^X:X\to K^q} K^q_{f_i^X} \xrightarrow{\text{project}} \prod_{q\in\mathbb{N}} \prod_{f_i^Y b:X\to K^q} K^q_{f_i^Y b} \tag{6.6}$$

and

$$T(X) \xrightarrow{P_{2X}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X:X\to PK^q} PK^q_{g_{i_j}^X} \xrightarrow{\text{project}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y b:X\to PK^q} PK^q_{g_{i_j}^Y b} \; . \tag{6.7}$$

We also have the maps

$$\prod_{q\in\mathbb{N}} \prod_{f_i^Y b:X\to K^q} K^q_{f_i^Y b} \longrightarrow \prod_{q\in\mathbb{N}} \prod_{f_i^Y:Y\to K^q} K^q_{f_i^Y} \tag{6.8}$$

and

$$\prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y b:X\to PK^q} PK^q_{g_{i_j}^Y b} \longrightarrow \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y:Y\to PK^q} PK^q_{g_{i_j}^Y} \; , \tag{6.9}$$

where the maps (6.8) and (6.9) are the identification of the corresponding factors. The composition of the maps (6.6) and (6.8) and the composition of the maps (6.7) and (6.9) can be completed to form the outer commutative square of the diagram

(6.10). The outer square clearly commutes as we are just taking subproducts of $T(X)$ indexed by maps factoring through Y and identifying with copies of K^q and PK^q indexed by those maps in the pullback of $T(Y)$.

$$
\begin{tikzcd}
T(X) \arrow[r, "P_{2X}"] \arrow[d, "P_{1X}"'] \arrow[dr, "T(b)"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g_{i_j}^X : X\to PK^q} PK^q_{g_{i_j}^X} \arrow[r, "\text{project}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g_{i_j}^Y b : X\to PK^q} PK^q_{g_{i_j}^Y b} \arrow[d, "\text{identify}"] \\
\prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^X : X\to K^q} K^q_{f_i^X} \arrow[d, "\text{project}"'] & T(Y) \arrow[r, "P_{2Y}"] \arrow[d, "P_{1Y}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g_{i_j}^Y : Y\to PK^q} PK^q_{g_{i_j}^Y} \arrow[d, "e"] \\
\prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^Y b : X\to K^q} K^q_{f_i^Y b} \arrow[r, "\text{identify}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^Y : Y\to K^q} K^q_{f_i^Y} \arrow[r, "\phi"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g_{i_j}^Y : Y\to PK^q} K^q_{g_{i_j}^Y}
\end{tikzcd}
\tag{6.10}
$$

Therefore, by the universal property of the pullback (cf.(2.12)), we get a unique map $T(b)$ from $T(X)$ to $T(Y)$.

6.1.5 The functor T

To show T is a functor, first we will prove the following lemma.

Lemma 6.1.3. *Let $b : X \longrightarrow Y$ and $a : Y \longrightarrow Z$ be maps of spaces X, Y and Z. Then $T(ab) = T(a)T(b)$.*

Proof. Let $f_i^Z : Z \longrightarrow K^q$ and $g_{i_j}^Z : Z \longrightarrow PK^q$, then by composing the maps we get $f_i^Z a : Y \longrightarrow Z \longrightarrow K^q$ and $g_{i_j}^Z a : Y \longrightarrow Z \longrightarrow PK^q$.

Given the maps $a : Y \longrightarrow Z$ and $b : X \longrightarrow Y$, using Section 6.1.4, we have $T(a) : T(Y) \longrightarrow T(Z)$ and $T(b) : T(X) \longrightarrow T(Y)$ respectively.
Furthermore using the composition $ab : X \longrightarrow Y \longrightarrow Z$ we can get $f_i^Z ab : X \longrightarrow Y \longrightarrow Z \longrightarrow K^q$ and $g_{i_j}^Z ab : X \longrightarrow Y \longrightarrow Z \longrightarrow PK^q$ and again as shown in

Section 6.1.4

$$T(X) \xrightarrow{P_{1X}} \prod_{q\in\mathbb{N}} \prod_{f_i^X : X\to K^q} K^q_{f_i^X} \xrightarrow{\text{project}} \prod_{q\in\mathbb{N}} \prod_{f_i^Z ab : X\to K^q} K^q_{f_i^Z ab} \xrightarrow{\text{identify}} \prod_{q\in\mathbb{N}} \prod_{f_i^Z : Z\to K^q} K^q_{f_i^Z} \tag{6.11}$$

and

$$T(X) \xrightarrow{P_{2X}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X : X\to PK^q} PK^q_{g_{i_j}^X} \xrightarrow{\text{project}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Z ab : X\to PK^q} PK^q_{g_{i_j}^Z ab} \xrightarrow{\text{identify}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Z : Z\to PK^q} PK^q_{g_{i_j}^Z} \tag{6.12}$$

implies a unique map $T(ab) : T(X) \longrightarrow T(Z)$.

(6.11) shows a map that takes $K^q_{f_i^Z ab} \subset T(X)$ indexed by a map $f_i^Z ab$, homeomorphically to $K^q_{f_i^Z} \subset T(Z)$. Similarly 6.12 is a map that takes $PK^q_{g_{i_j}^Z ab} \subset T(X)$ indexed by a map $g_{i_j}^Z ab$, homeomorphically to $PK^q_{g_{i_j}^Z} \subset T(Z)$.

But $T(ab) : T(X) \longrightarrow T(Z)$ can also be factored as

$$\begin{tikzcd}
T(X) \arrow[r, "P_{1X}"] \arrow[dr, bend right, "T(b)"'] & \displaystyle\prod_{q\in\mathbb{N}} \prod_{f_i^X : X\to K^q} K^q_{f_i^X} \arrow[r, "\text{project}"] & \displaystyle\prod_{q\in\mathbb{N}} \prod_{f_i^Y b : X\to K^q} K^q_{f_i^Y b} \arrow[d, "\text{identify}"] & \\
& T(Y) \arrow[r, "P_{1X}"] \arrow[dr, bend right, "T(a)"'] & \displaystyle\prod_{q\in\mathbb{N}} \prod_{f_i^Y : Y\to K^q} K^q_{f_i^Y} \arrow[r, "\text{project}"] & \displaystyle\prod_{q\in\mathbb{N}} \prod_{f_i^Z a : Y\to K^q} K^q_{f_i^Z a} \arrow[d, "\text{identify}"] \\
& & T(Z) \arrow[r, "P_{1X}"] & \displaystyle\prod_{q\in\mathbb{N}} \prod_{f_i^Z : Z\to K^q} K^q_{f_i^Z}
\end{tikzcd} \tag{6.13}$$

and (6.12) can be factored as

$$
\begin{tikzcd}
T(X) \arrow[r, "P_{2X}"] \arrow[dr, bend right, "T(b)"'] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^X_{i_j}:X\to PK^q} PK^q_{g^X_{i_j}} \arrow[r, "\text{project}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^Y_{i_j}b:X\to PK^q} PK^q_{g^Y_{i_j}b} \arrow[d, "\text{identify}"] & \\
 & T(Y) \arrow[r, "P_{2X}"] \arrow[dr, bend right, "T(a)"'] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^Y_{i_j}:Y\to PK^q} PK^q_{g^Y_{i_j}} \arrow[r, "\text{project}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^Z_{i_j}a:Y\to PK^q} PK^q_{g^Z_{i_j}a} \arrow[d, "\text{identify}"] \\
 & & T(Z) \arrow[r, "P_{2X}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^Z_{i_j}:Z\to PK^q} PK^q_{g^Z_{i_j}}
\end{tikzcd}
\tag{6.14}
$$

The two diagrams (6.13) and (6.14) explains the composition $T(a)T(b)$.

$T(a)$ is basically a projection out to factors of $T(Y)$ indexed by $f_i^Z a$ and $g_{i_j}^Z a$, similarly $T(b)$ is a projection out to factors of $T(Y)$ indexed by $f_i^Y b$ and $g_{i_j}^Y b$. Therefore the composition $T(a)T(b)$ is a projection out to factors of $T(X)$ indexed by $f_i^Z ab$ and $g_{i_j}^Z ab$.

Hence we have $T(ab) = T(a)T(b)$. □

Lemma 6.1.4. *Let $id_X : X \longrightarrow X$ be the identity map on X and $1_{T(X)} : T(X) \longrightarrow T(X)$ be the identity map on $T(X)$ then $T(id_X) = 1_{T(X)}$.*

Proof. Let $f_i^X : X \longrightarrow K^q$ and $g_{i_j}^X : X \longrightarrow PK^q$. The maps $f_i^X = f_i^X id_X : X \longrightarrow X \longrightarrow K^q$ and $g_{i_j}^X = g_{i_j}^X id_X : X \longrightarrow X \longrightarrow PK^q$ are used to define $T(id_X) : T(X) \longrightarrow T(X)$, by Section 6.1.4. It is clear that $T(id_X)$ takes all the factors of $T(X)$ identically to $T(X)$.
Therefore, $T(1_X) = 1_{T(X)}$. □

Using Lemma 6.1.3 and Lemma 6.1.4 we have shown the construction T on spaces is a functor, $T : \mathcal{CW}_* \longrightarrow \mathcal{CW}_*$.

Let $T^1(X) := T(X)$, and iterating this functorial construction on spaces n-times, where $n \in \mathbb{N}$, we get the space $T^n(X) := T(T^{n-1}(X))$, for $n > 1$. There are two natural maps $\varsigma_X : X \longrightarrow T(X)$ and $\beta_X : T^2(X) \longrightarrow T(X)$, associated with this construction, which we will explain now.

6.1.6 Unit ς_X

Given a space X, and for every map $f : X \longrightarrow K^q$ there are canonical maps $\{f_i^X\} : X \longrightarrow \prod_{q\in\mathbb{N}} \prod_{f_i^X : X\to K^q} K^q_{f_i^X}$ such that $f = pr_f\{f_i^X\}$, where $pr_f : \prod_{q\in\mathbb{N}} \prod_{f_i^X : X\to K^q} K^q_{f_i^X} \longrightarrow K^q_f$ is the canonical projection. Similarly, for every $g : X \longrightarrow PK^q$, there are canonical maps $\{g_{i_j}^X\} : X \longrightarrow \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X : X\to PK^q} PK^q_{g_{i_j}^X}$ such that $g = pr_g\{g_{i_j}^X\}$.

$$\begin{tikzcd}
X \arrow[dr, dotted, "\varsigma_X"] \arrow[drr, bend left, "\{g_{i_j}^X\}"] \arrow[ddr, bend right, "\{f_i^X\}"'] & & \\
& T(X) \arrow[r, "P_{2X}"] \arrow[d, "P_{1X}"'] & \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X : X\to PK^q} PK^q_{g_{i_j}^X} \arrow[d, "e"] \\
& \prod_{q\in\mathbb{N}} \prod_{f_i^X : X\to K^q} K^q_{f_i^X} \arrow[r, "\phi"'] & \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^X : X\to PK^q} K^q_{g_{i_j}^X}
\end{tikzcd} \tag{6.15}$$

The image of ϕ are products of $K^q_{g_{i_j}^X}$, which are end points of the path space as explained in section 6.1 therefore the outer square commutes. Then by the universal property of the pullback the map $\varsigma_X : X \longrightarrow T(X)$ is the unique map into the pullback.

Lemma 6.1.5. *$\varsigma : I \longrightarrow T$, defined by ς_X for each $X \in \mathcal{CW}_*$ is a natural transformation.*

Proof. To show $\varsigma : I \longrightarrow T$ is a natural transformation we need to show the square (6.16) commutes for any $b : X \longrightarrow Y$ in $\mathcal{CW}_*$.

$$\begin{tikzcd}
I(X) \arrow[r, "\varsigma_X"] \arrow[d, "I(b)"'] & T(X) \arrow[d, "T(b)"] \\
I(Y) \arrow[r, "\varsigma_Y"] & T(Y)
\end{tikzcd} \tag{6.16}$$

Given $b : X \longrightarrow Y$, from Section 6.1.4 we have $T(b) : T(X) \longrightarrow T(Y)$.

$$X \xrightarrow{\{f_i^Y b\}} \prod_{q\in\mathbb{N}} \prod_{f_i^Y b : X \to K^q} K^q_{f_i^Y b} \xrightarrow{\text{identify}} \prod_{q\in\mathbb{N}} \prod_{f_i^Y : Y \to K^q} K^q_{f_i^Y} \tag{6.17}$$

and

$$X \xrightarrow{\{g_{i_j}^Y b\}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y b : X \to PK^q} PK^q_{g_{i_j}^Y b} \xrightarrow{\text{identify}} \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y : Y \to PK^q} PK^q_{g_{i_j}^Y} \tag{6.18}$$

combined with the universal property of the pullback gives the map $T(b)\varsigma_X$: $I(X) \longrightarrow T(X) \longrightarrow T(Y)$.

The maps $\{f_i^Y\}b : X \longrightarrow Y \longrightarrow \prod_{q\in\mathbb{N}} \prod_{f_i^Y : Y \to K^q} K^q_{f_i^Y}$ and $\{g_{i_j}^Y\}b : X \longrightarrow Y \longrightarrow \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^Y : Y \to PK^q} PK^q_{g_{i_j}^Y}$ together with the universal property of the pullback defines $\varsigma_Y I(b) : I(X) \longrightarrow I(Y) \longrightarrow T(Y)$. Since $\{f_i^Y b\} = \{f_i^Y\}b$ and $\{g_{i_j}^Y b\} = \{g_{i_j}^Y\}b$, we have $\varsigma_Y b = T(b)\varsigma_X$. $\square$

6.1.7 Multiplication β_X

For any space X, $T^2(X)$ is the pullback shown in the following diagram.

$$\begin{tikzcd} T^2(X) \arrow[r, "P_{2TX}"] \arrow[d, "P_{1TX}"'] & \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^{TX} : T(X) \to PK^q} PK^q_{g_{i_j}^{TX}} \arrow[d, "e"] \\ \prod_{q\in\mathbb{N}} \prod_{f_i^{TX} : T(X) \to K^q} K^q_{f_i^{TX}} \arrow[r, "\phi"'] & \prod_{q\in\mathbb{N}} \prod_{g_{i_j}^{TX} : T(X) \to PK^q} K^q_{ev\, g_{i_j}^{TX}} \end{tikzcd} \tag{6.19}$$

If $f : X \longrightarrow K^q$, then there is a factor of K^q_f in $\prod_{q\in\mathbb{N}} \prod_{f_i^X : X \to K^q} K^q_{f_i^X}$ and we have the map

$$pr_f P_{1X} : T(X) \xrightarrow{P_{1X}} \prod_{q\in\mathbb{N}} \prod_{f_i^X : X \to K^q} K^q_{f_i^X} \xrightarrow{pr_f} K^q_f \, . \tag{6.20}$$

Hence there will be a factor of $K^q_{pr_f P_{1X}}$ in $\displaystyle\prod_{q\in\mathbb{N}}\prod_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}}$. Therefore we have the map,

$$pr_{(pr_f P_{1X})}P_{1TX} : T^2(X) \xrightarrow{P_{1TX}} \prod_{q\in\mathbb{N}}\prod_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}} \xrightarrow{pr_{(pr_f P_{1X})}} K^q_{pr_f P_{1X}}\,. \tag{6.21}$$

From the space $\displaystyle\prod_{q\in\mathbb{N}}\prod_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}}$ we project to factors of the form $\displaystyle\prod_{q\in\mathbb{N}}\prod K^q_{pr_f P_{1X}}$ and then identify with factors of the form $\displaystyle\prod_{q\in\mathbb{N}}\prod_{f_i^X:X\to K^q} K^q_{f_i^X}$

So we have a map $\displaystyle\prod_{q\in\mathbb{N}}\prod_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}} \longrightarrow \prod_{q\in\mathbb{N}}\prod_{f_i^X:X\to K^q} K^q_{f_i^X}$.

Using a similar argument, if $g : X \longrightarrow PK^q$ it can be shown that there is a map $\displaystyle\prod_{q\in\mathbb{N}}\prod_{g_{i_j}^{TX}:T(X)\to PK^q} PK^q_{g_{i_j}^{TX}} \longrightarrow \prod_{q\in\mathbb{N}}\prod_{g_{i_j}^X:X\to PK^q} PK^q_{g_{i_j}^X}$ where we identify the factor of $PK^q_{pr_g P_{2X}}$ in $\displaystyle\prod_{q\in\mathbb{N}}\prod_{g_{i_j}^{TX}:T(X)\to PK^q} PK^q_{g_{i_j}^{TX}}$ with PK^q_g in $\displaystyle\prod_{q\in\mathbb{N}}\prod_{g_{i_j}^X:X\to PK^q} PK^q_{g_{i_j}^X}$.

$$\begin{tikzcd}
T^2(X) \arrow[r, "P_{2TX}"] \arrow[d, "P_{1TX}"'] \arrow[dr, dotted, "\beta_X"] & \displaystyle\prod_{q\in\mathbb{N}}\prod_{g_{i_j}^{TX}:T(X)\to PK^q} PK^q_{g_{i_j}^{TX}} \arrow[r, "\text{project}"] & \displaystyle\prod_{q\in\mathbb{N}}\prod PK^q_{pr_g P_{2X}} \arrow[d, "\text{identify}"] \\
\displaystyle\prod_{q\in\mathbb{N}}\prod_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}} \arrow[d, "\text{project}"] & T(X) \arrow[r, "P_{2X}"] \arrow[d, "P_{1X}"'] & \displaystyle\prod_{q\in\mathbb{N}}\prod_{g_{i_j}^X:X\to PK^q} PK^q_{g_{i_j}^X} \arrow[d, "e"] \\
\displaystyle\prod_{q\in\mathbb{N}}\prod K^q_{pr_f P_{1X}} \arrow[r, "\text{identify}"] & \displaystyle\prod_{q\in\mathbb{N}}\prod_{f_i^X:X\to K^q} K^q_{f_i^X} \arrow[r, "\phi"'] & \displaystyle\prod_{q\in\mathbb{N}}\prod_{g:X\to PK^q} K^q_{ev\,g}
\end{tikzcd} \tag{6.22}$$

The outer square in the diagram (6.22) commutes because the inner square of diagram (6.22) and diagram (6.19) commute. Therefore by the universal property of the pullback there exist a unique map $\beta_X : T^2(X) \longrightarrow T(X)$.

Lemma 6.1.6. *$\beta : T^2 \longrightarrow T$, defined by β_X for each $X \in \mathcal{CW}_*$ is a natural transformation.*

Proof. To show $\beta : T^2 \longrightarrow T$ is a natural transformation we need to show the following square commutes for any $b : X \longrightarrow Y$ in $\mathcal{CW}_*$.

$$\begin{tikzcd}
T^2(X) \arrow[r, "\beta_X"] \arrow[d, "T^2(a)"'] & T(X) \arrow[d, "T(a)"] \\
T^2(Y) \arrow[r, "\beta_Y"] & T(Y)
\end{tikzcd}$$

The map $T(a)\beta_X : T^2(X) \longrightarrow T(Y)$ factors through $T(X)$ as shown in Figure (6.1).

$$\begin{tikzcd}
T^2(X) \arrow[r, "P_{2TX}"] \arrow[d, "P_{1TX}"] \arrow[dr, "\beta_X"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{TX}_{i_j}:T(X)\to PK^q} PK^q_{g^{TX}_{i_j}} \arrow[r, "\text{project}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{pr_f P_{2X}:TX\to PK^q} PK^q_{pr_f P_{2X}} \arrow[d, "\text{identify}"] & & \\
\prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{TX}_{i}:T(X)\to K^q} K^q_{f^{TX}_i} \arrow[d, "\text{project}"] & T(X) \arrow[r, "P_{2X}"] \arrow[d, "P_{1X}"] \arrow[drr, "T(a)"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{X}_{i_j}:X\to PK^q} PK^q_{g^{X}_{i_j}} \arrow[rr, "\text{project}"] & & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{Y}_{i_j}a:X\to PK^q} PK^q_{g^{Y}_{i_j}a} \arrow[d, "\text{identify}"] \\
\prod\limits_{q\in\mathbb{N}} \prod\limits_{pr_f P_{1X}:TX\to K^q} K^q_{pr_f P_{1X}} \arrow[r, "\text{identify}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{X}_{i}:X\to K^q} K^q_{f^{X}_i} \arrow[d, "\text{project}"] & & T(Y) \arrow[r, "P_{2Y}"] \arrow[d, "P_{1Y}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{Y}_{i_j}:Y\to PK^q} PK^q_{g^{Y}_{i_j}} \arrow[d, "e"] \\
& \prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{Y}_{i}a:X\to K^q} K^q_{f^{Y}_i a} \arrow[rr, "\text{identify}"] & & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{Y}_{i}:Y\to K^q} K^q_{f^{Y}_i} \arrow[r, "\phi"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{X}_{i_j}:X\to PK^q} K^q_{ev\, g^{X}_{i_j}}
\end{tikzcd}$$

Figure 6.1

Let γ_1 be the map identify $pr_{f^Y_i a}\, P_{1X} : TX \longrightarrow K^q_{f^Y_i}$ and γ_2 be the map identify $pr_{g^Y_{i_j} a}\, P_{2X} : TX \longrightarrow PK^q_{g^Y_{i_j}}$. (6.1) shows, maps that take factors $K^q_{\gamma_1}$ and $PK^q_{\gamma_2}$ in $T^2(X)$

indexed by maps γ_1 and γ_2 homeomorphically to $K^q_{f_i^Y}$ and $PK^q_{g^Y_{i_j}}$ in $T(Y)$ respectively.

$$\begin{tikzcd}
T^2(X) \arrow[r, "P_{2TX}"] \arrow[d, "P_{1TX}"] \arrow[dr, "T^2(a)"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{TX}_{i_j}:T(X)\to PK^q} PK^q_{g^{TX}_{i_j}} \arrow[r, "\text{project}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{TX}_{i_j}T(a):T(X)\to PK^q} PK^q_{g^{TX}_{i_j}T(a)} \arrow[d, "\text{identify}"'] & \\
\prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{TX}_i:T(X)\to K^q} K^q_{f^{TX}_i} \arrow[d, "\text{project}"] & T^2(Y) \arrow[r, "P_{2TY}"] \arrow[d, "P_{1TY}"'] \arrow[dr, "\beta_Y"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{TY}_{i_j}:TY\to PK^q} PK^q_{g^{TY}_{i_j}} \arrow[r, "\text{project}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{pr_f P_{2Y}:Y\to PK^q} PK^q_{pr_f P_{2Y}} \arrow[d, "\text{identify}"'] \\
\prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{TY}_i T(a):T(X)\to K^q} K^q_{f^{TY}_i T(a)} \arrow[r, "\text{identify}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{TY}_i:TY\to K^q} K^q_{f^{TY}_i} \arrow[d, "\text{project}"] & T(Y) \arrow[r, "P_{2Y}"] \arrow[d, "P_{1Y}"'] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{Y}_{i_j}:Y\to PK^q} PK^q_{g^Y_{i_j}} \arrow[d, "e"'] \\
 & \prod\limits_{q\in\mathbb{N}} \prod\limits_{pr_f P_{1Y}:Y\to K^q} K^q_{pr_f P_{1Y}} \arrow[r, "\text{identify}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{Y}_i:Y\to K^q} K^q_{f^Y_i} \arrow[r, "\phi"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{X}_{i_j}:X\to PK^q} K^q_{ev\, g^X_{i_j}}
\end{tikzcd}$$

Figure 6.2

Let $pr_{f_i^Y} P_{1Y} T(a) : T(X) \longrightarrow K^q_{f_i^Y}$, then (6.2) shows maps that take factors $K^q_{pr_{f_i^Y} P_{1Y} T(a)}$ in $T^2(X)$ homeomorphically to $K^q_{f_i^Y}$ in $T(Y)$.
It is clear that both the maps $pr_{f_i^Y} P_{1Y} T(a)$ and identify $pr_{f_i^Y a}\, P_{1X}$ take the factors $K^q_{f_i^Y a}$ to $K^q_{f_i^Y}$. So $pr_{f_i^Y} P_{1Y} T(a) = \text{identify}\, pr_{f_i^Y a}\, P_{1X}$ index the same factor of $K^q \subset T^2(X)$ that is mapped homeomorphically to $K^q_{f_i^Y}$. Similarly $pr_{g^Y_{i_j}} P_{1Y} T(a) = \text{identify}\, pr_{g^Y_{i_j} a}\, P_{2X}$ Therefore $\beta_X T(a) = T^2(a)\beta_Y$. $\square$

6.2 The monad (T, ς, β)

The triple (T, ς, β) forms a monad on spaces, which we verify shortly. The triple (T, ς, β) is a monad if it satisfies the two diagrams given in (2.27) and (2.26).

Lemma 6.2.1. *For each $X \in \mathcal{CW}_*$*

(i) $\beta_X \varsigma_{T(X)} = 1_{T(X)}$

(ii) $\beta_X T(\varsigma_X) = 1_{T(X)}$

Proof. First we will prove $\beta_X \varsigma_{T(X)} = 1_{T(X)}$

Let $f : X \longrightarrow K^q$, then we have the map

$$pr_f P_{1X} : T(X) \xrightarrow{P_{1X}} \prod_{q \in \mathbb{N}} \prod_{f_i^X : X \to K^q} K^q_{f_i^X} \xrightarrow{pr_f} K^q_f$$

as explained in (6.20). Using the universal property of the product there exists a unique map $\{pr_f P_{1X}\} : T(X) \longrightarrow \prod_{q \in \mathbb{N}} \prod_{f_i^{TX} : X \to K^q} K^q_{f_i^{TX}}$ into the subproduct indexed by projection.

$$\begin{tikzcd}
K^q_{pr_f P_{1X}} & & \prod_{q \in \mathbb{N}} \prod_{f_i^{TX} : X \to K^q} K^q_{f_i^{TX}} \arrow[ll] \\
 & & T(X) \arrow[ull, "pr_f P_{1X}"] \arrow[u, "\{pr_f P_{1X}\}"']
\end{tikzcd} \tag{6.23}$$

As β_X projects onto factors that are indexed by projections $\beta_X \varsigma_{T(X)}$ is equivalent to $\beta_X \{pr_f P_{1X}\}$. As $\varsigma_{T(X)}$ takes K^q_f in $T(X)$ identically to the factor indexed by $K^q_{pr_f P_{1X}}$ in $T^2(X)$, we have $\beta_X \varsigma_{T(X)}$ takes K^q_f in $T(X)$ identically to factors indexed by K^q_f in $T(X)$. Similarly, PK^q_g in $T(X)$ is taken to PK^q_g in $T(X)$. Hence $\beta_X \varsigma_{T(X)} = 1_{T(X)}$.

(ii) Now we need to show, $\beta_X T(\varsigma_X) = 1_{T(X)}$. We define $T(\varsigma_X)$ as explained in Section 2.1.2 (cf.2.1). We apply the functor T on the map $\varsigma_X : X \longrightarrow TX$, to get $T\varsigma_X = T(\varsigma_X)$ as shown in the diagram (6.24).

$$\begin{tikzcd} T(X) \arrow[r, "T"] & T^2(X) \\ X \arrow[u, "\varsigma_X"] \arrow[r, "T"'] & T(X) \arrow[u, "T(\varsigma_X)"'] \end{tikzcd} \tag{6.24}$$

From Section 6.1.4 we know how T acts on maps, so we have

$$\prod_{q\in\mathbb{N}} \prod_{f_i^X : X \to K^q} K^q_{f_i^X} \xrightarrow{\text{project}} \prod_{q\in\mathbb{N}} \prod_{f_i^{TX}\varsigma_X : X \to K^q} K^q_{f_i^{TX}\varsigma_X} \xrightarrow{\text{identify}} \prod_{q\in\mathbb{N}} \prod_{f_i^{TX} : T(X) \to K^q} K^q_{f_i^{TX}} \tag{6.25}$$

Therefore $T(\varsigma_X)$ takes $K^q_{f_i^{TX}\varsigma_X}$ in $T(X)$ to $K^q_{f_i^{TX}}$ in $T^2(X)$.

For every $f : X \longrightarrow K^q$ as in Section 6.1.7 there is a factor of $K^q_f \subset \prod_{q\in\mathbb{N}} \prod_{f_i^X : X \to K^q} K^q_{f_i^X}$ and a factor of $K^q_{pr_f P_{1X}} \subset \prod_{q\in\mathbb{N}} \prod_{f_i^{TX} : T(X) \to K^q} K^q_{f_i^{TX}}$ and from the map (6.25) we get

$$\prod_{q\in\mathbb{N}} \prod_{f : X \to K^q} K^q_f \xrightarrow{\text{project}} \prod_{q\in\mathbb{N}} \prod_{pr_f P_{1X}\,\varsigma_X : X \to K^q} K^q_{pr_f P_{1X}\,\varsigma_X} \xrightarrow{\text{identify}} \prod_{q\in\mathbb{N}} \prod_{pr_f P_{1X} : T(X) \to K^q} K^q_{pr_f P_{1X}} \tag{6.26}$$

This can also be explained by a diagram as follows

$$
\begin{tikzcd}
 & & & K^q_{pr_f P_{1X}} \\
K^q_f & T(X) \arrow[l, "pr_f P_{1X}"'] \arrow[r, "T"] & T^2(X) \arrow[r, "P_{1TX}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}} \arrow[u, "pr_{f_i^{TX}}"] \\
 & & & \prod\limits_{q\in\mathbb{N}} \prod\limits_{pr_f P_{1X}\varsigma_X : X\to K^q} K^q_{pr_f P_{1X}\varsigma_X} \arrow[u, "\text{identify}"'] \\
 & X \arrow[uul, "f"] \arrow[uu, "\varsigma_X"'] \arrow[r, "T"] & T(X) \arrow[uu, "T(\varsigma_X)"'] \arrow[r, "P_{1X}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^X : X\to K^q} K^q_{f_i^X} \arrow[u, "\text{project}"'] \arrow[d, "pr_{f_i^{TX}}\varsigma_X"'] \\
 & & & K^q_{pr_f P_{1X}\varsigma_X} \arrow[uuuu, bend right, "id_{K^q}"']
\end{tikzcd}
\tag{6.27}
$$

Therefore $T(\varsigma_X)$ identifies $K^q_f \subset T(X)$ with the factor of $K^q_{pr_f P_{1X}} \subset T^2(X)$ which β_X then identifies back with K^q_f. It is clear that a similar argument for path spaces holds. $\square$

Now we proceed to show the associative diagram (2.26) of a monad holds.

Lemma 6.2.2. *For each $X \in \mathcal{CW}_*$ $\beta_X \beta_{T(X)} = \beta_X(T\beta_X)$.*

Proof. Given $\beta_X : T^2(X) \longrightarrow T(X)$, we take the functor T on this map to get the commutative square (6.28).

$$
\begin{tikzcd}
T^2(X) \arrow[r, "T"'] \arrow[d, "\beta_X"'] & T^3(X) \arrow[d, "T(\beta_X)"] \\
T(X) \arrow[r, "T"] & T^2(X)
\end{tikzcd}
\tag{6.28}
$$

If $f : X \longrightarrow K^q$, then we have the map $pr_f P_{1TX} : T(X) \longrightarrow K^q$ by (6.20) and also the map $pr_{pr_f P_{1X}} P_{1TX} : T^2(X) \longrightarrow K^q$ by (6.21). So the map $pr_f P_{1TX}$ indexes a factor of $K^q_{pr_f P_{1TX}}$ in $\prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}}$ and the map $pr_{pr_f P_{1TX}} P_{1T^2X}$ indexes a factor of $K^q_{pr_{pr_f P_{1TX}} P_{1T^2X}}$ in $\prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{T^2X}:T^2(X)\to K^q} K^q_{f_i^{T^2X}}$. Using Section 2.1.2 we define $T\beta_X$ as $T(\beta_X)$.

$$\begin{tikzcd}
 & & & K^q_{pr_{pr_f P_{1X}} P_{1TX}} \arrow[dddd, bend left=90, "id_{K^q}"] \\
 & T^2(X) \arrow[r, "T"] \arrow[ddl, "pr_{pr_f P_{1X}} P_{1TX}"'] \arrow[dd, "\beta_X"'] & T^3(X) \arrow[r, "P_{1T^2X}"] \arrow[dd, "T(\beta_X)"'] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{T^2X}:T^2(X)\to K^q} K^q_{f_i^{T^2X}} \arrow[u, "pr_{pr_f P_{1X}} P_{1TX}"] \arrow[d, "\text{project}"'] \\
 & & & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{TX}\beta_X:X\to K^q} K^q_{f_i^{TX}\beta_X} \arrow[d, "\text{identify}"'] \\
K^q_{pr_f P_{1X}} & T(X) \arrow[l, "pr_f P_{1X}"] \arrow[r, "T"] & T^2(X) \arrow[r, "P_{1TX}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{TX}:T(X)\to K^q} K^q_{f_i^{TX}} \arrow[d, "pr_f P_{1X}"'] \\
 & & & K^q_{pr_f P_{1X}}
\end{tikzcd} \tag{6.29}$$

Then $T(\beta_X)$ takes the factor of $K^q_{pr_{pr_f P_{1TX}} P_{1T^2X}}$ to the factor of $K^q_{pr_f P_{1TX}}$.

$$\begin{tikzcd}
K^q_{pr_{pr_f P_{1TX}} P_{1T^2X}} \arrow[d, "T(\beta_X)"'] & \subset T^3(X) \\
K^q_{pr_f P_{1TX}} \arrow[d, "\beta_X"'] & \subset T^2(X) \\
K^q_f & \subset T(X)
\end{tikzcd} \tag{6.30}$$

It is clear that $\beta_X \beta_{T(X)}$ takes $K^q_{pr_{pr_f P_{1X}} P_{1TX}} \mapsto K^q_{pr_f P_{1X}} \mapsto K^q_f$. A similar argument shows diagram (2.26) holds for path spaces. Hence the associative square (2.26) for the monad commutes. □

6.3 Cosimplicial resolution

Given $X \in \mathcal{CW}_*$, and the monad (T, ς, β) of Section 6.2 we will follow the method explained in Section 2.2.3 of constructing a cosimplicial object in $\mathcal{CW}_*$ from a monad. The monad (T, ς, β) generates a cosimplicial functor $(C^n, \delta^i_n, \sigma^i_n)_{n\geq 0}$, where we define

$$\begin{aligned} C^n(X) &:= T^{n+1}(X)\,, n \geq 0 \\ \delta^i_{nX} &: C^{n-1}(X) \longrightarrow C^n(X) \text{ where, } \delta^i_n := T^i \varsigma T^{n-i}, \text{ and } 0 \leq i \leq n \\ \sigma^i_{nX} &: C^{n+1}(X) \longrightarrow C^n(X) \text{ where, } \sigma^i_n := T^i \beta T^{n-i}\ . \text{ and } 0 \leq i \leq n \end{aligned}$$

We apply the cosimplicial functor $(C^n, \delta^i_n, \sigma^i_n)_{n\geq 0}$ to the space X, to get the cosimplicial object $C^\bullet X$ coaugmented by X as in diagram (6.31), where $\varsigma_X : X \longrightarrow C^0(X)$ is the coaugmentation (Remark 2.2.14).

Notation 6.3.1. *To simplify the notation we will write δ^i_n and σ^i_n for δ^i_{nX} and σ^i_{nX} respectively.*

$$X \xrightarrow{\varsigma_X} C^0(X) \underset{\xrightarrow{\delta^1_1}}{\xrightarrow{\delta^0_1}} \xleftarrow{\sigma^0_0} C^1(X) \begin{smallmatrix} \xrightarrow{\delta^0_2} \\ \xrightarrow{\delta^1_2} \\ \xrightarrow{\delta^2_2} \end{smallmatrix} \begin{smallmatrix} \xleftarrow{\sigma^1_1} \\ \xleftarrow{\sigma^0_1} \end{smallmatrix} C^2(X) \cdots \tag{6.31}$$

We then take the q^{th} cohomology functor on the cosimplicial space (6.31) to obtain $H^q(X) \xleftarrow{d_{aug}} H^q(C^\bullet X)$, which is a simplicial abelian group augmented by $H^q(X)$. The arrows are reversed because the functor H^q is contravariant

$$H^q(X) \xleftarrow{d_{aug}} H^q(C^0(X)) \xrightarrow{s^0_0} \begin{smallmatrix} \xleftarrow{d^1_0} \\ \xleftarrow{d^1_1} \end{smallmatrix} H^q(C^1(X)) \begin{smallmatrix} \xrightarrow{s^1_1} \\ \xrightarrow{s^0_1} \end{smallmatrix} \begin{smallmatrix} \xleftarrow{d^2_0} \\ \xleftarrow{d^2_1} \\ \xleftarrow{d^2_2} \end{smallmatrix} H^q(C^2(X)) \cdots \tag{6.32}$$

The face map d_i^n and the degeneracy map s_i^n in the simplicial abelian group $H^q(C^\bullet X)$ are induced by δ_n^i and σ_n^i respectively. The augmentation map d_{aug} in the simplicial abelian group $H^q(C^\bullet X)$ is induced by ς_X. Using (2.2.19) we get the Moore chain complex

$$H^q(C^0(X)) \xleftarrow{d_0^1} H^q(C^1(X)) \cap \ker d_1^1 \xleftarrow{d_0^2} H^q(C^2(X)) \cap \ker d_1^2 \cap \ker d_2^2 \cdots \tag{6.33}$$

In Theorem 6.3.2, we will show the chain complex (6.33) is acyclic

Theorem 6.3.2. *Let $\Phi \in H^q(C^{p-1}(X))$, $p > 1$ be such that $d_0^{p-1}(\Phi) = d_1^{p-1}(\Phi) = \cdots = d_{p-1}^{p-1}(\Phi) = 0 \in H^q(C^{p-2}(X))$. Then there exists $\gamma \in H^q(C^p(X))$ such that $d_0^p(\gamma) = \Phi$ and $d_1^p(\gamma) = d_2^p(\gamma) = \cdots = d_p^p(\gamma) = 0$.*

Proof. We choose $f : C^{p-1}(X) \longrightarrow K^q$ such that f is a map representing the class $\Phi \in H^q(C^{p-1}(X))$. Then we let γ represent the class $[pr_f P_{1T^p(X)}]$, where $pr_f P_{1T^p(X)} : C^p(X) \longrightarrow K^q$.

(i) First, we will show $d_0^p(\gamma) = \Phi$. Consider the map $\varsigma_{T^p(X)} : C^{p-1}(X) \longrightarrow TC^{p-1}(X)$, defined by

$$\begin{tikzcd} C^{p-1}(X) \arrow[r, "\varsigma_{T^p(X)}"] \arrow[drr, bend right, "f"'] & TC^{p-1}(X) \arrow[r, "P_{1T^p(X)}"] & \prod_{q\in\mathbb{N}} \prod_{f_i^{T^p(X)}:T^p(X)\to K^q} K^q_{f_i^{T^p(X)}} \arrow[d, "pr_f"] \\ & & K^q \end{tikzcd} \tag{6.34}$$

and a similar map into the product of path spaces.

The map $\delta_p^0 = \varsigma_{T^p(X)}$ induces the map d_0^p, so we get

$$\begin{aligned} d_0^p(\gamma) &= d_0^p[pr_f P_{1T^p(X)}] \\ &= [pr_f P_{1T^p(X)}\, \delta_p^0] \\ &= [pr_f P_{1T^p(X)}\, \varsigma_{T^p(X)}] \\ &= [f] \\ &= \Phi. \end{aligned}$$

(ii) To show $d_1^p(\gamma) = d_2^p(\gamma) = \cdots = d_p^p(\gamma) = 0$. Let $1 \leq j \leq p$ then $0 \leq j-1 \leq p-1$.

Given the maps $\delta_{p-1}^{j-1} : C^{p-2}(X) \longrightarrow C^{p-1}(X)$ and $f : C^{p-1}(X) \longrightarrow K^q$, we have the following well-defined composition,

$$f\delta_{p-1}^{j-1} : C^{p-2}(X) \xrightarrow{\delta_{p-1}^{j-1}} C^{p-1}(X) \xrightarrow{f} K^q\ . \tag{6.35}$$

Since $f\delta_{p-1}^{j-1}$ is a map from $C^{p-2}(X) \longrightarrow K^q$, there is a factor of $K^q_{f\delta_{p-1}^{j-1}}$ in $\prod_{n\in\mathbb{N}} \prod_{f_i^{T^{p-1}(X)}:T^{p-1}(X)\to K^q} K^q_{f_i^{T^{p-1}(X)}}$ and hence a map $pr_{f\delta_{p-1}^{j-1}} P_{1T^{p-1}(X)} : C^{p-1}(X) \longrightarrow K^q_{f\delta_{p-1}^{j-1}}$.

From Section 2.1.2 , for $j \geq 1$, we have

$$\begin{aligned} \delta_p^j &= (T^j \varsigma T^{p-j})_X \\ &= (TT^{j-1}\varsigma T^{p-j})_X \\ &= T(T^{j-1}\varsigma T^{(p-1)-(j-1)})_X \\ &= T\delta_{p-1}^{j-1} \end{aligned}$$

From Section 6.1.4 we know how functor T act on the map $\delta_{p-1}^{j-1} : C^{p-2}(X) \longrightarrow C^{p-1}(X)$, also using a similar argument as in diagram (6.27) we get the following diagram

$$
\begin{tikzcd}
 & & & K^q_f \\
K^q & C^{p-1}(X) \arrow[l, "f"'] \arrow[r, "T"] & TC^{p-1}X \arrow[r, "P_{1T^pX}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{T^pX}:T^p(X)\to K^q} K^q_{f_i^{T^pX}} \arrow[u, "pr_f"] \\
 & & & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{T^pX}\delta^{j-1}_{p-1}:T^{p-1}(X)\to K^q} K^q_{f_i^{T^pX}\delta^{j-1}_{p-1}} \arrow[u, "\text{identify}"'] \\
 & C^{p-2}(X) \arrow[uul, "f\delta^{j-1}_{p-1}"] \arrow[uu, "\delta^{j-1}_{p-1}"'] \arrow[r, "T"] & TC^{p-2}(X) \arrow[uu, "T\delta^{j-1}_{p-1}"] \arrow[r, "P_{1T^{p-1}X}"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f_i^{T^{p-1}X}:T^{p-1}(X)\to K^q} K^q_{f_i^{T^{p-1}X}} \arrow[u, "\text{project}"'] \arrow[d, "pr_{f\delta^{j-1}_{p-1}}"] \\
 & & & K^q_{f\delta^{j-1}_{p-1}} \arrow[uuuu, bend right=80, "id_{K^q}"']
\end{tikzcd}
\tag{6.36}
$$

Now $T\delta^{j-1}_{p-1}$ takes the factor $K^q_{f\delta^{j-1}_{p-1}}$ to the factor K^q_f.

Therefore, chasing the diagram (6.36) we have

$$
\begin{aligned}
pr_f P_{1T^p(X)} T\delta^{j-1}_{p-1} &= pr_{f\delta^{j-1}_{p-1}} P_{1T^{p-1}(X)} && (6.37)\\
\text{which implies } pr_f P_{1T^p(X)} \delta^{j}_{p} &= pr_{f\delta^{j-1}_{p-1}} P_{1T^{p-1}(X)} && (6.38)\\
\text{So } d^p_j[pr_f P_{1T^p(X)}] &= [pr_{f\delta^{j-1}_{p-1}} P_{1T^{p-1}(X)}]\,. && (6.39)
\end{aligned}
$$

Since by our hypothesis $d^{p-1}_{j-1}(\Phi) = f\delta^{j-1}_{p-1}$ is null-homotopic, then the map $pr_{f\delta^{j-1}_{p-1}} P_{1T^{p-1}(X)}$:

$T^p(X) \longrightarrow K^q_{f\delta^{j-1}_{p-1}}$ factors through the path space as in (6.3).

$$\begin{tikzcd} & T^p(X) \arrow[r, "P_{2T^{p-1}X}"] \arrow[d, "P_{1T^{p-1}X}"'] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{T^{p-1}X}_{i_j}:T^{p-1}X\to PK^q} PK^q_{g^{T^{p-1}X}_{i_j}} \arrow[d, "e"] \\ & \prod\limits_{q\in\mathbb{N}} \prod\limits_{f^{T^{p-1}X}_{i}:T^{p-1}X\to K^q} K^q_{f^{T^{p-1}X}_{i}} \arrow[r, "\phi"] & \prod\limits_{q\in\mathbb{N}} \prod\limits_{g^{T^{p-1}X}_{i_j}:X\to PK^q} K^q_{g^{T^{p-1}X}_{i_j}} \end{tikzcd} \tag{6.40}$$

So there is an homeomorphic map factoring through the path space and therefore each $[pr_{f\delta^{j-1}_{p-1}} P_{1T^{p-1}(X)}] = d^p_j(\gamma)$ (in (6.39)) factors through the path space $PK^q_{f\delta^{j-1}_{p-1}{}_j}$, hence it is null-homotopic.

By (i) and (ii) we have $\ker d^{p-1}_0 = \operatorname{im} d^p_0$. □

Corollary 6.3.3. *Let $d^0_0 = d_{aug}$, then*
$H^q(X) \overset{d^0_0}{\longleftarrow} H^q(C^0(X)) \overset{d^1_0}{\longleftarrow} H^q(C^1(X)) \cap \ker d^1_1$ is exact.

Proof. The same argument given in the proof of (i) in Theorem 6.3.2 can be used to show that if $\Phi \in H^q(C^0(X))$ is such that $d^0_0(\Phi) = 0 \in H^q(X)$ then there exists $\gamma \in H^q(C^1(X))$ such that $d^1_0(\gamma) = [pr_f P_{1T(X)} \varsigma_{TX}] = [f] = \Phi$.

In the proof of Lemma 6.2.1 (ii) we have defined $T\varsigma_X$. Then letting $\delta^0_0 = \varsigma_X$ and using a similar argument as in the proof of Theorem 6.3.2 (ii) it can be easily verifed that $d^1_1(\gamma) = 0$. Therefore $\ker d_{aug} = \ker d^0_0 = \operatorname{im} d^1_0$, hence exact at $H^q(C^0(X))$. □

Theorem 6.3.4.

(I) $\pi_p H^q(C^\bullet X) = 0$ *if* $p \geq 1$

(II) $\pi_0 H^q(C^\bullet X) = H^q(X)$ *is an isomorphism for all* $q \geq 1$

Proof. Let $[\Phi] \in H^q(X)$ be the class representing $f : X \longrightarrow K^q$, since d_{aug} is induced by the map $\varsigma_X : X \longrightarrow TX$. The diagram is commutative

$$\begin{tikzcd} X \arrow[r, "\varsigma_X"] \arrow[dr, bend right, "f"'] & T(X) \arrow[r, "P_{1X}"] & \prod_{q\in\mathbb{N}} \prod_{f_i^X : X \to K^q} K^q_{f_i^X} \arrow[d, "pr_f"] \\ & & K^q_f \end{tikzcd} \tag{6.41}$$

Therefore $f = pr_f P_{1X} \varsigma_X$ which implies $[f] = d_{aug}[pr_f P_{1X}]$. Therefore d_{aug} is onto. From homological algebra we know ([49]) if d_{aug} in $0 \leftarrow H^q(X) \xleftarrow{d_{aug}} H^q(C^0(X))$ is onto then we have a short exact sequence

$$0 \leftarrow H^q(X) \xleftarrow{d_{aug}} H^q(C^0(X)) \leftarrow \ker d_{aug} \leftarrow 0$$

which splits as

$$H^q(X) \cong H^q(C^0(X))/\ker d_{aug} = H^q(C^0(X))/\mathrm{im}\, d_0^1$$

Since $\pi_0 H^q(C^\bullet X) = H^q(C^0(X))/\mathrm{im}\, d_0^1$, we have

(I) $\pi_p H^q(C^\bullet X) = 0$ if $q \geq 1$

(II) $\pi_0 H^q(C^\bullet X) = H^q(X)$

Note: d_{aug} onto in $0 \leftarrow H^q(X) \xleftarrow{d_{aug}} H^q(C^0(X))$ also implies this sequence is exact at $H^q(X)$. Then by Theorem 6.3.2 and Corollary 6.3.3 the augmented Moore chain complex for (6.32) is exact, so we could use Fact 2.2.18 to deduce the same conclusion. □

Theorem 6.3.5. *$\pi_0 H^*(C^\bullet X) = H^*(X)$.*

Proof. Let $C^\bullet X$ be a cosimplicial space (6.31). We take the functor $[C^\bullet X, \quad] : \mathcal{H}(\mathcal{R}) \longrightarrow \mathcal{SET}_*$ as shown in Figure 6.3. In Figure 6.3 each column is a simplicial abelian group and each row is an $\mathcal{H}(\mathcal{R})$-algebra.

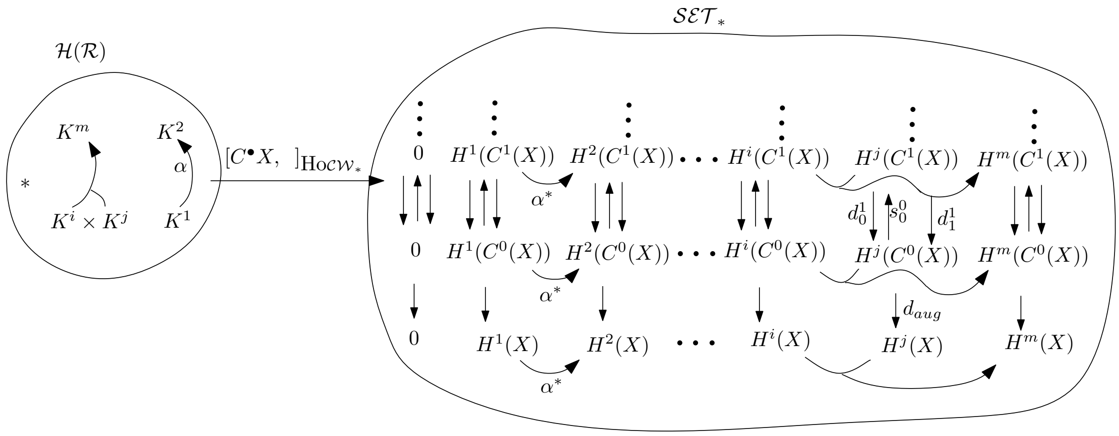

Figure 6.3

From Theorem 6.3.4 we have $\pi_0 H^q(C^\bullet X) \cong H^q(X)$. For any cohomology operation $\alpha : \prod_{i \in J} K^{n_i} \longrightarrow K^m$, there is an induced map $\alpha_X : \prod_{i \in J} H^{n_i}(X) \longrightarrow H^m(X)$. We also have the isomorphism $d_{aug} : \prod_{i \in J} H^{n_i}(C^\bullet X)/\mathrm{ker} d_{aug} = \prod_{i \in J} \pi_0 H^{n_i}(C^\bullet X) \longrightarrow \prod_{i \in J} H^{n_i}(X)$. Since d_{aug} is a natural transformation we have the commutative square

$$\begin{tikzcd}
\prod_{i\in J} H^{n_i}(C^\bullet X)/\mathrm{ker} d_{aug} \arrow[d, equal] & \prod_{i\in J} H^{m}(C^\bullet X)/\mathrm{ker} d_{aug} \arrow[d, equal] \\
\prod_{i\in J} \pi_0 H^{n_i}(C^\bullet X) \arrow[r, "\pi_0 \alpha_{C^\bullet X}"] \arrow[d, "d_{aug}"'] & \prod_{i\in J} \pi_0 H^{m}(C^\bullet X) \arrow[d, "d_{aug}"] \\
\prod_{i\in J} H^{n_i}(X) \arrow[r, "\alpha_X"'] & H^m(X)
\end{tikzcd} \tag{6.42}$$

Therefore it is clear that $\mathcal{H}(\mathcal{R})$-algebra structure of $H^*(X)$ is the same as the $\mathcal{H}(\mathcal{R})$-algebra structure of $\pi_0 H^*(C^\bullet X)$ $\square$

Chapter 7

Application and future work

7.1 Problems with the construction $T(X)$ for $\mathcal{R} = \mathbb{Z}$

Our construction $T(X)$ in Section 6.1 is dual to Stover's construction $\mathcal{V}(X)$ ([53], 2.2). Then in ([53], 2.3 and 2.4), Stover constructs a simplicial space, $V_\bullet X$, from a $\mathcal{CW}_*$ space X, where $V_\bullet X$ is homotopy equivalent to a wedge of spheres in each simplicial dimension. Taking the p-th homotopy group of this simplicial space gives a simplicial group $\pi_p(V_\bullet X)$ and the homotopy groups of these simplicial groups satisfy

$$\pi_q \pi_p(V_\bullet X) = 0 \quad \text{for all} \quad q \geq 1 \quad \text{and} \quad p \geq 1$$

$$\pi_0 \pi_p(V_\bullet X) = \pi_p(X)$$

Zisman's ([18], Appendix) modification of the Bousfield-Friedlander spectral sequence has $E^2_{p,q} = \pi_q \pi_p X_\bullet$ and this spectral sequence converges strongly to $\pi_{p+q}(|X_\bullet|)$ (where $|X_\bullet|$ denotes the realization of the simplicial space $X_\bullet$). Therefore Stover's resolution fit into the E^2 page of the Zisman's Bousfield-Friedlander spectral sequence and he could conclude

$$E^2_{p,q} = \pi_q \pi_p V_\bullet X \Rightarrow \pi_{p+q}(|V_\bullet X|)$$

Using ([53], 2.6 and 3.4) Stover shows in ([53], 3.5) there is a homotopy equivalence between $|V_\bullet X|$ and X and this allowed him to identify the E^2 terms of the spectral sequence with his resolution converging to the Π-algebra $\pi_*(X)$.

It was hoped the dual Stover construction $T(X)$ would provide better understanding of the algebra of integral cohomology operations including the unstable compositions and torsion cross cap products ([46] $\oint 3$) and allow more powerful tools incoporating this structure.

We identify two main problems associated with $\mathbb{Z}$ coefficients with our work in contrast to Stover.

1. From Theorem 6.3.4 we have

(a) $\pi_p H^q(C^\bullet X) = 0$ if $p \geq 1$

(b) $\pi_0 H^q(C^\bullet X) = H^q(X)$ is an isomorphism for all $q \geq 1$

Dual to Stover we would like to use a cohomology spectral sequence (Dwyer Spectral sequence [21]) with $E^2_{p,q} = \pi_p H^q(X^\bullet)$ which converges to $H^{p+q}(\mathrm{Tot}(X^\bullet))$. However, this spectral sequence is not known to converge for $\mathcal{R} = \mathbb{Z}$, even when $\mathcal{R} = \mathbb{F}_p$ it has some convergence issues.

2. The second point to note is that $\mathrm{Tot}(X^\bullet)$ may not have the same cohomology type as X or the $\mathcal{R}$-completion $R_\infty X$.

Fact 7.1.1. Bousfield in ([15], 7.5) has shown that given a monad $(\top, \eta, \mu)$ and $X \in \mathrm{Ho}(\mathcal{C})$ such that $\top$ preserves weak equivalences and has the further properties

(a) $\top X$ is a group object in $\mathrm{Ho}(\mathcal{C})$

(b) $\Omega\top X$ is $\top$-injective in $\mathrm{Ho}(\mathcal{C})$

then the monad $(\top, \eta, \mu)$ fits into his framework ([15], 7.5) and the cosimplicial space $X^\bullet$ coming from this monad gives rise to a homotopy spectral sequence ([15], 5.8) which converges to $\pi_{q-p}(\mathrm{Tot}(X^\bullet))$. He observes that in ([15], 4.9 and 7.7), only when $\mathcal{G}$ contains Eilenberg-Mac Lane spaces over all $\mathcal{R}$-modules (not just Eilenberg-Mac Lane spaces over $\mathcal{R}$ itself) will $\mathrm{Tot}(X^\bullet) \cong R_\infty X$. This means resolutions must be acyclic for cohomology in all $\mathcal{R}$-module coefficients.

Our monad (T, ς, β) of Section 6.2 is only acyclic over a fixed ring $\mathcal{R}$. So for the cosimplicial resolution $X \longrightarrow T^\bullet X$ we may not have $\mathrm{Tot}(X^\bullet) \cong R_\infty X$.

This suggests that we could modify our definition for $\mathcal{H}(\mathbb{Z})$ to contain products and loops of Eilenberg-Mac Lane spaces over all $\mathbb{Z}$-modules, but then we run into set-theoretic complications. According to Mac Lane ([41] page 23) the category $\mathcal{R} - \mathrm{mod}$ is a large category. We want to modify $\mathcal{H}(\mathcal{R})$ to contain arbitrary products formed from a proper class, but forming products over a proper class is not defined according to ([43], pg 108).

7.2 The construction $T(X)$ for $\mathcal{R} = \mathbb{F}_p$

Although $\mathrm{Tot}(X^\bullet)$ may not be R-equivalent to $R_\infty X$ for our cosimplicial resolution with coefficients in an arbitrary ring $\mathcal{R}$ as explained in Section 7.1, this is not the case for $\mathcal{R} = \mathbb{F}_p$. From Example 3.4.9 we know that $\mathcal{H}(\mathbb{F}_p)$-algebras are the algebras over the Steenrod algebras and it turns out that $\mathcal{H}(\mathbb{F}_p)$-algebras are more nicely behaved as compared to $\mathcal{H}(\mathbb{Z})$-algebras.

From algebra we know that any $\mathbb{F}_p$-module (a vector space) is a direct sum of copies of $\mathbb{F}_p$. For a cosimplicial resolution $X \longrightarrow X^\bullet$ to be acyclic over $\mathbb{F}_p$ implies the resolution is acyclic over all $\mathbb{F}_p$-vector spaces ([17], $\oint$ 11) because they are a direct sum of copies of $\mathbb{F}_p$. Consequently we have $\mathrm{Tot}(T^\bullet X) \cong R_\infty X$.

7.3 Homology spectral sequence for a cosimplicial space

We will use homology instead of cohomology because the homology spectral sequence for a cosimplicial space is known to converge strongly for field coefficients under certain conditions given below [14]. To use the homology spectral sequence we will need to go from cohomology to homology and for this we will need to impose some finiteness conditions on the space X. The main reason we need these conditions is due to Fact 7.3.3, but first we give some definitions.

Definition 7.3.1. *A space X is of finite type ([16], V 7.5) if $H_n(X; \mathcal{R})$ is finitely generated for all $n \geq 0$.*

Definition 7.3.2. *A space X is finite if X has a finite number of finitely generated homology groups.*

Fact 7.3.3. From algebra we know that a finite dimensional vector space has the same dimension as its algebraic dual, therefore both the vector space and its algebraic dual have a finite basis. This result does not hold for an infinite dimensional vector space.

Remark 7.3.4. *For a space X of finite type the graded vector space $H^n(X; \mathbb{F}_p)$ is isomorphic to its algebraic dual $H_n(X; \mathbb{F}_p) \cong Hom(H^n(X; \mathbb{F}_p); \mathbb{F}_p)$. This can also be derived from the universal coefficient theorem. That is, for the exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow Hom(H_n(X), G) \longrightarrow 0 \qquad (7.1)$$

the Ext *term vanishes for field coefficients giving the required ismorphism* $H_n(X;\mathbb{F}_p) \cong Hom(H^n(X;\mathbb{F}_p);\mathbb{F}_p)$ *([30], 3.1).*

The $\mathcal{H}(\mathbb{F}_p)$*-algebra (Steenrod algebra) structure is still preserved because the Steenrod algebra has a dual action on homology giving a coalgebra over the Steenrod algebra [44].*

Definition 7.3.5. *A Generalized Eilenberg-Mac Lane space (denoted by* $\mathcal{R}$*-GEM) is a space homotopy equivalent to* $\prod_{i\in I} K(A_i, n_i)$ *with* A_i *an* $\mathcal{R}$*-module and* I *an indexing set ([20], ∮ 5).*

Definition 7.3.6. *A connected space* X *is called nilpotent ([51], ∮ 5) if its fundamental group acts nilpotently on each* $\pi_i(X)$ *for* $i \geq 1$*. A connected space* X *is called* p*-nilpotent it it is nilpotent and* $\pi_i(X)$ *is a* p*-group with bounded torsion for each* i*.*

Shipley [51] generalizes the convergence conditions for the homology spectral sequence developed by Bousfield in [14] from earlier work ([17], ∮ 10). In Shipley's result the space $\mathrm{Tot}(X^\bullet)$ is required to be p-good where as in Bousfield's result ([14], 3.6) he needed $\mathrm{Tot}(X^\bullet)$ to be simply connected.

Fact 7.3.7. (Shipley [51], ∮ 2 and Theorem 6.1) Let $X^\bullet$ be a fibrant cosimplicial space with each X^s p-nilpotent and X^s of finite type for each s. Assume either

(a) $\mathrm{Tot}(X^\bullet)$ is of finite type or

(b) $\varprojlim H_*\mathrm{Tot}_s(X^\bullet)$ is finitely generated

Then the homology spectral sequence for $X^\bullet$ has $E^2_{s,t} = \pi^s H_t(X^\bullet;\mathbb{F}_p)$ and this spectral sequence converges strongly to $H_*(\mathrm{Tot}(X^\bullet))$ if and only if $\mathrm{Tot}(X^\bullet)$ is p-good.

Example 7.3.8. Let X be a simply connected space. If there are null-homotopic maps $X \longrightarrow K(\mathbb{F}_p, 2)$ a factor of $PK(\mathbb{F}_p, 2) \cong \Omega K(\mathbb{F}_p, 2) \cong K(\mathbb{F}_p, 1)$ is used in the construction of $T(X)$. The space $T(X)$ may not be nilpotent without more restrictions on X. This is because the construction $T(X)$ becomes more and more connected at each simplicial dimension.
Note: $T^2(X)$ onwards is simply connected.

Remark 7.3.9. *The cosimplicial resolution $T^\bullet X$ is generally infinite and not nilpotent, therefore the resolution does not directly satisfy the conditions of the homology spectral sequence (7.3.7). However, because $T^\bullet X$ is $\mathcal{G}$-equivalent to $R^\bullet X$ for $\mathcal{G}$ the set of Eilenberg-Mac Lane spaces over all $\mathbb{F}_p$ vector spaces in the resolution model category, the E^2-term gives an isomorphism $\pi^s H_t(R^\bullet X; \mathbb{F}_p) \cong \pi^s H_t(T^\bullet X; \mathbb{F}_p)$. With $\pi^s H_t(R^\bullet X; \mathbb{F}_p)$ converging to $H_*(R_\infty X; \mathbb{F}_p) \cong H_*(X; \mathbb{F}_p)$ ([16], I 5.1 5.2 and III 5.4), so $R^\bullet X$ can be replaced with $T^\bullet X$ and we have $\pi^s H_t(T^\bullet X; \mathbb{F}_p) \Rightarrow H_*(X; \mathbb{F}_p)$.*

7.4 Mapping space

Definition 7.4.1. *Let X and Y be pointed spaces then the mapping space* $\mathrm{map}_*(Y, X)$ *is the space of continuous maps from Y to X with the compact open topology.*

Fact 7.4.2. (Shipley [51], Theorem 6.2) Let X and Y be spaces such that Y is finite and X is of finite type. Assume either

(a) $\mathrm{map}_*(Y, R_\infty X)$ is of finite type.

(b) $\varprojlim H_* \,\mathrm{map}_*(Y, \mathrm{Tot}_s R^\bullet X)$ is finitely generated

Then the homology spectral sequence for $\mathrm{map}_*(Y, R^\bullet X)$ strongly converges to $H_* \,\mathrm{map}_*(Y, R_\infty X)$ if and only if $\mathrm{map}_*(Y, R_\infty X)$ is p-good.

Remark 7.4.3. *Similar to Remark 7.3.9, because we have a $\mathcal{G}$-equivalence between the cosimplicial resolutions $R^\bullet X$ and $T^\bullet X$ giving an isomorphism of the E^2-term of the homology spectral sequence applied to the mapping space, we have $\pi^s H_t(\mathrm{map}_*(Y, R^\bullet X); \mathbb{F}_p) \cong \pi^s H_t(\mathrm{map}_*(Y, T^\bullet X); \mathbb{F}_p)$. Then by Fact 7.4.2 the spectral sequence for the mapping space with $E^2_{s,t} = \pi^s H_t(\mathrm{map}_*(Y, R^\bullet X); \mathbb{F}_p)$ converges to $H_*(\mathrm{map}_*(Y, R_\infty X); \mathbb{F}_p)$. Replacing $R^\bullet X$ with $T^\bullet X$ we have*
$E^2_{s,t} = \pi^s H_t(\mathrm{map}_*(Y, T^\bullet X); \mathbb{F}_p)$ converges to $H_*(\mathrm{map}_*(Y, R_\infty X); \mathbb{F}_p)$.

Example 7.4.4. If $Y = S^1$, then $\mathrm{map}_*(S^1, T^\bullet X) \cong \Omega T^\bullet X$. So we get the mapping space spectral sequence converging to $H_*(\mathrm{map}_*(S^1, R_\infty X); \mathbb{F}_p) \cong H_*(\Omega R_\infty X; \mathbb{F}_p)$.

7.5 Future work

1. In Definition 3.3.2 we have defined abstract $H(\mathbb{Z})$-algebras, as product preserving functors from $\mathcal{H}(\mathbb{Z})$ to $\mathcal{SET}_*$. The realization problem can be stated as: Which abstract $H(\mathbb{Z})$-algebras can be realized as a cohomology $H(\mathbb{Z})$-algebra? The realization problem for $H(\mathbb{F}_p)$-algebras, that is, which algebras over the Steenrod algebra can be realized, were studied by many and solved for special cases [3, 38, 19, 4]. We can ask whether dual methods to those used to realize Π-algebras [7, 10] can be used for both $\mathbb{Z}$ and $\mathbb{F}_p$ coefficients.

2. Can the conditions be found, so the cohomology spectral sequence with $E_2^{p,q} = \pi_p H^q(X^\bullet; \mathbb{Z})$ converges to $H^*(\mathrm{Tot}(X^\bullet); \mathbb{Z})$ and can this be used to show $\mathrm{Tot}X^\bullet$ has the same cohomology as the augmentation $H^*(X; \mathbb{Z})$.

3. It is known that under certain conditions the cohomology spectral sequence of a cosimplicial space converges for $\mathbb{F}_p$ coefficients [21, 11]. By studying the conditions for which the spectral sequence converges, can we get results directly

using the cohomology spectral sequence without having to use dual vector spaces and homology spectral sequence?

4. As discussed in Section 7.1, for the $Tot(X^\bullet) \cong R_\infty X$ we need $\mathcal{G}$ to be Eilenberg-Mac Lane spaces over all $\mathbb{Z}$-modules. Can Bousfield's conditions ([15], 7.5) be relaxed so that the resolutions being acyclic to smaller test set still works? For example, $\mathcal{G}$ containing products of Eilenberg-Mac Lane spaces over finitely generated abelian groups.

5. How can we interpret $\pi^s H_t(\mathrm{map}_*(Y, T^\bullet X); \mathbb{F}_p)$ as a derived functor? In the diagram of categories and functors in Figure 7.1 , we know the functor $[\mathrm{map}_*(Y, T^\bullet X), \ \ ] : \mathcal{H}(\mathbb{F}_p) \longrightarrow s\mathbb{F}_p\text{-VEC}$ is a covariant functor and $Hom(\ \ , \mathbb{F}_p)$ is a contravariant functor.

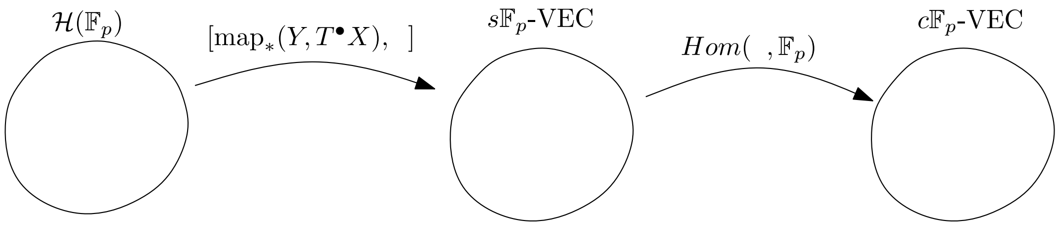

Figure 7.1

Therefore the composition $Hom([\mathrm{map}_*(Y, T^\bullet X), \ \ ], \mathbb{F}_p)$ is contravariant and hence sends products in $\mathcal{H}(\mathbb{F}_p)$ to coproducts in $c\mathbb{F}_p$-VEC. The image of the functor $Hom([\mathrm{map}_*(Y, T^\bullet X), \ \ ], \mathbb{F}_p)$ is $H_*(\mathrm{map}_*(Y, T^\bullet X); \mathbb{F}_p)$. Another question we can ask that is related to this is, what is the categorical setting for the co-model objects of a sketch category? Or equivalently how can we encode the homology operation structure as a natural transformation?

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