Stability of Stationary Solutions to Curvature Flows

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Abstract

In this thesis we study the evolution of hypersurfaces under weighted volume preserving curvature flows. Specifically we consider the stability of spheres and finite cylinders as stationary solutions to the flows. The flows are formulated as a partial differential equation for a height function and an existence result is obtained when the height function is small. Through further analysis we prove that the sphere and finite cylinder, provided the radius of the finite cylinder satisfies a certain condition, are stable. That is, we prove that if a graph over a sphere or cylinder has small height function its flow exists for all time and converges to a sphere or cylinder respectively. This is the first result proving that there exist non-axially symmetric hypersurfaces that converge to cylinders under the flows.

In the case of volume preserving mean curvature flow near a cylinder, we improve the above results to obtain greater regularity of the flow and convergence with respect to a stricter norm. Analysing the condition on the radius in this situation we find it is necessary in order for the cylinder to be stable. The analysis also leads to the surprising result that certain constant mean curvature unduloids are stable stationary solutions to the axially symmetric flow in high dimensions. The last result of the thesis proves the instability of two dimensional catenoids under the classical mean curvature flow.

The results in this thesis are obtained using functional analysis and semigroup methods, which can be applied since the linearised speed operators are sectorial. The stability results come from analysing the spectrum of the linearised operators and analysing the center manifold of the system.

Declaration

I herewith declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other institution and affirms that to the best of the candidate's knowledge the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

David Hartley,

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1

Introduction

1.1 Background

The mean curvature flow (MCF) evolves a hypersurface over time with speed that at each point is given by its mean curvature and the direction is along the unit normal. The flow was first studied in a geometric measure theory setting by Brakke in [16]. If we consider an embedding of the hypersurface $\boldsymbol{X}_0 : M^n \to \mathbb{R}^{n+1}$ then the flow is equivalent to solving the partial differential equation (PDE) for a family of embeddings $\boldsymbol{X} : M^n \times [0, T) \to \mathbb{R}^{n+1}$:

$$\frac{\partial \boldsymbol{X}}{\partial t} = -H\boldsymbol{\nu}, \quad \boldsymbol{X}(\cdot, 0) = \boldsymbol{X}_0, \tag{1.1}$$

where the mean curvature, H, is given by the sum of the principal curvatures, κ_a, of the hypersurface $\Omega_t := \boldsymbol{X}(M^n, t)$ and $\boldsymbol{\nu}$ is a choice of unit normal of Ω_t.

This flow has been extensively studied, with many results relating to the asymptotic behaviour of the hypersurfaces and formulation of singularities. A classic paper by Huisken [30] proved that uniformly convex hypersurfaces under MCF, with $n \geq 2$, will shrink to a point in a finite time while becoming asymptotically spherical. This means that, after a rescaling to preserve area and to have the flow exist for all time, the flow converges to a sphere. This result has been expanded on by Gage and Hamilton [23] who proved the analogous case with $n = 1$. Grayson [25] showed that plane curves will become convex before they become singular. This leads to the remarkable result that any smooth, closed, compact, plane-embedded curve will shrink to a point in a finite time while asymptotically becoming a circle. Ecker and Huisken, [18], expanded their research to include non-compact hypersurfaces and proved long time existence for the flow where the initial hypersurface is an entire graph over the plane and satisfies a gradient bound. They also prove that if the initial hypersurface satisfies a linear growth condition then the hypersurface becomes asymptotically selfsimilar. Results relating to singularities can be found in [1, 8, 32, 33, 43] for example.

A related problem is the volume preserving mean curvature flow, where a forcing term is added to the PDE so that an enclosed volume relating to the hypersurface is constant throughout the flow:

$$\frac{\partial \boldsymbol{X}}{\partial t} = \left(\fint_{M^n} H \, d\mu - H \right) \boldsymbol{\nu}, \qquad \boldsymbol{X}(\cdot, 0) = \boldsymbol{X}_0, \tag{1.2}$$

where $d\mu$ is the induced measure on Ω_t. This flow was first studied for hypersurfaces by Huisken in [31] where he proved that initially convex hypersurfaces have a flow that exists for all time and converges to a sphere as $t \to \infty$. For non-convex, compact, closed hypersurfaces Escher and Simonett in [21] proved that if a hypersurface is a graph over a sphere with height function sufficiently small, then under the flow it will converge to a sphere. A similar result was obtained by Li [36] where, instead of having small height function, the hypersurface was average mean convex with small traceless second fundamental form. Athanassenas and Kandanaarachchi, [12], make use of axial symmetry to remove any conditions on curvature and obtain convergence to spheres under the assumption no singularities develop on the axis of rotation.

The case where the initial hypersurface has a boundary has also been studied. In this case it is assumed that Ω_0 is smoothly embedded in the domain

$$W = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} : 0 < x_{n+1} < d \right\},$$

with $d > 0$ and $\partial\Omega_0 \subset \partial W$. The open set enclosed by Ω_0 and W will be labelled Φ and it is the volume of Φ that is preserved under the flow. The boundary conditions for the flow are that Ω_t meets ∂W orthogonally. Assuming Ω_0 to be axially symmetric it was proved in [11] that the flow exists for all time and converges to a cylinder in W of volume $Vol(\Phi)$, under the assumption

$$|\Omega_0| \leq \frac{Vol(\Phi)}{d}. \tag{1.3}$$

This constraint ensures that the solution never touches the axis of rotation, so that no singularities develop.

The volume preserving mean curvature flow can be generalised to the weighted volume preserving curvature flows. These flows evolve a hypersurface over time by a symmetric function of the principal curvatures, along with a global forcing term. The PDE that represents the flow is given by:

$$\frac{\partial \boldsymbol{X}}{\partial t} = \left(\frac{1}{\int_{M^n} \Xi\left(\boldsymbol{\kappa}\right) \, d\mu} \int_{M^n} F\left(\boldsymbol{\kappa}\right) \Xi\left(\boldsymbol{\kappa}\right) \, d\mu - F\left(\boldsymbol{\kappa}\right) \right) \boldsymbol{\nu}, \qquad \boldsymbol{X}\left(\cdot, 0\right) = \boldsymbol{X}_0, \tag{1.4}$$

where $F\left(\boldsymbol{\kappa}\right) = F\left(\kappa_1, \ldots, \kappa_n\right)$ and $\Xi\left(\boldsymbol{\kappa}\right) = \Xi\left(\kappa_1, \ldots, \kappa_n\right)$ are smooth, symmetric functions of the principal curvatures, κ_a, of Ω_t. Note that we must restrict to initial hypersurfaces such that $\int_{M^n} \Xi\left(\boldsymbol{\kappa}\right) \, d\mu > 0$.

When $\Xi = E_a$, an elementary symmetric function of the principal curvatures (see (2.1)), the flow is the mixed volume preserving curvature flow and preserves a certain quantity of the hypersurface (see Corollary 2.2.3). This flow has been previously studied by McCoy in [40] where he proved that under some additional assumptions on F, for example strict positivity, homogeneity of degree one, convexity and increasing on the positive cone, strictly convex hypersurfaces have a flow that exists for all time and the hypersurfaces converge to a sphere as $t \to \infty$. This was an extension of [39], where he proved the result under the condition that $F = H$. Volume preserving flows, $\Xi \equiv 1$, have been been studied by Cabezas-Rivas and Sinestrari in [17] for the case where F is a power of the m^{th} mean curvature, $H_m = \binom{n}{m}^{-1} E_m$. The flow was shown to take initially convex hypersurfaces that satisfy the pinching condition $E_n > CH^n$, for a specific constant C, to spheres.

Throughout this thesis we will consider the case where the initial embedding is a normal graph over another hypersurface, i.e. $\boldsymbol{X}_{\rho_0}\left(\boldsymbol{p}\right) = \boldsymbol{X}_0\left(\boldsymbol{p}\right) + \rho_0\left(\boldsymbol{p}\right) \boldsymbol{\nu}_0\left(\boldsymbol{p}\right)$ for $\boldsymbol{p} \in M^n$, where we now define $\boldsymbol{X}_0$ to be an embedding of the base hypersurface and $\boldsymbol{\nu}_0$ is a normal to the base hypersurface. In this case the flow (1.4) is equivalent, up to a tangential diffeomorphism (see Lemma 2.3.3), to the PDE:

$$\frac{\partial \rho}{\partial t} = \sqrt{1 + \left|\tilde{\nabla} \rho\right|^2} \left(\frac{1}{\int_{M^n} \Xi\left(\boldsymbol{\kappa}_\rho\right) \, d\mu_\rho} \int_{M^n} F\left(\boldsymbol{\kappa}_\rho\right) \Xi\left(\boldsymbol{\kappa}_\rho\right) \, d\mu_\rho - F\left(\boldsymbol{\kappa}_\rho\right) \right), \quad \rho\left(\cdot, 0\right) = \rho_0, \tag{1.5}$$

where we use a ρ subscript to show the dependence of quantities on the height function. The quantity $\sqrt{1 + \left|\tilde{\nabla} \rho\right|^2}$, see Section 1.3 for a definition of $\left|\tilde{\nabla} \rho\right|$, is similar to the gradient function used in [18] as it is the inverse of the inner product between the normals of $\Omega_\rho := \boldsymbol{X}_\rho\left(M^n\right)$ and $\Omega_0 := \boldsymbol{X}_0\left(M^n\right)$ (see Lemma 2.3.1). When the base hypersurface has a boundary we use the Neumann boundary condition $\nabla\rho|_{\partial M^n} \cdot \boldsymbol{\upsilon} = 0$, where $\boldsymbol{\upsilon}$ and $\boldsymbol{\nu}$ form an orthonormal basis for the normal space of ∂M^n, see Figure 1.1 for a graph over a cylinder. This boundary condition is natural as it is known that critical points to the area functional under a volume constraint necessarily satisfy it, [9, 14].

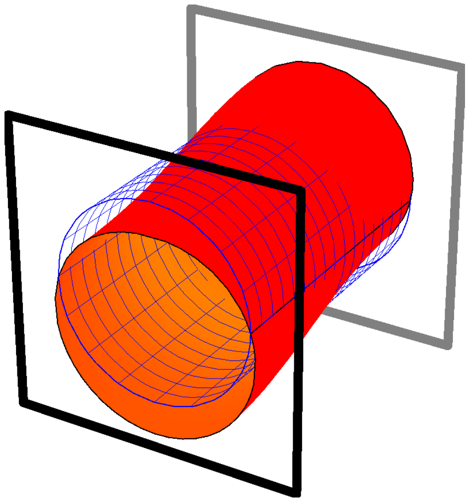


Figure 1.1: A graph over a cylinder satisfying the Neumann boundary condition

We also have the following assumptions on F and Ξ:

(A1) F and Ξ are smooth, symmetric functions

(A2) $\frac{\partial F}{\partial \kappa_a}(\boldsymbol{\kappa}_0) > 0$ for every $a = 1, \ldots, n$

(A3) $\Xi(\boldsymbol{\kappa}_0) > 0$.

The conditions (A1) and (A2) ensure isotropicity and parabolicity of the flow, respectively, while condition (A3) ensures there exists a neighbourhood of zero such that $\int_{M^n} \Xi(\boldsymbol{\kappa}_\rho)\, d\mu_\rho > 0$ for any ρ in this neighbourhood. We again have the classical volume preserving mean curvature flow for $F(\boldsymbol{\kappa}_\rho) = H$ and $\Xi(\boldsymbol{\kappa}_\rho) = 1$:

$$\frac{\partial \rho}{\partial t} = \sqrt{1 + \left|\tilde{\nabla}\rho\right|^2} \left(\fint_{M^n} H(\rho)\, d\mu_\rho - H(\rho) \right), \qquad \rho(\cdot, 0) = \rho_0. \tag{1.6}$$

In this thesis we will consider the stability of the sphere of radius R, $\mathscr{S}^n_R$, and the cylinder of radius R and length d, $\mathscr{C}^n_{R,d}$, under the weighted volume preserving curvature flows, as well as the stability of catenoids under mean curvature flow. In the cases of the cylinder and catenoid the presence of a boundary can cause difficulties in the analysis. In the case of a cylindrical graph, we set up a related PDE on the torus with one flat direction $\mathscr{T}^n_{R,d} = \mathscr{S}^{n-1}_R \times \mathscr{S}^1_{\frac{d}{\pi}}$ to overcome these difficulties. That is, we extend the metric, g, and the second fundamental form, A, of the cylinder evenly so that they are symmetric $(0,2)$-forms on the torus. Note that the former becomes the metric on $\mathscr{T}^n_{R,d}$. We can then use the formulas in Section 2.3 to define the operator $\boldsymbol{\kappa}_u$ and volume form $d\mu_u$ abstractly for a function u on $\mathscr{T}^n_{R,d}$, which replaces ρ as our 'height' function, and consider the PDE:

$$\frac{\partial u}{\partial t} = \sqrt{1 + \left|\tilde{\nabla} u\right|^2} \left(\frac{1}{\int_{\mathscr{T}^n_{R,d}} \Xi\left(\boldsymbol{\kappa}_u\right) \, d\mu_u} \int_{\mathscr{T}^n_{R,d}} F\left(\boldsymbol{\kappa}_u\right) \Xi\left(\boldsymbol{\kappa}_u\right) \, d\mu_u - F\left(\boldsymbol{\kappa}_u\right) \right), \; u\left(\cdot, 0\right) = u_0. \tag{1.7}$$

In the case that u is an even function $\boldsymbol{\kappa}_u$ and $d\mu_u$ preserve this symmetry, therefore the speed operator will also preserve the symmetry. Hence, any solution to (1.7) where u_0 is even will remain even for all time and will therefore satisfy $\nabla u|_{\partial \mathscr{C}^n_{R,d}} \cdot \boldsymbol{v} = 0$, whenever u is differentiable. Further, if u is even we have that:

$$\frac{1}{\int_{\mathscr{T}^n_{R,d}} \Xi\left(\boldsymbol{\kappa}_u\right) \, d\mu_u} \int_{\mathscr{T}^n_{R,d}} F\left(\boldsymbol{\kappa}_u\right) \Xi\left(\boldsymbol{\kappa}_u\right) \, d\mu_u = \frac{1}{\int_{\mathscr{C}^n_{R,d}} \Xi\left(\boldsymbol{\kappa}_u\right) \, d\mu_u} \int_{\mathscr{C}^n_{R,d}} F\left(\boldsymbol{\kappa}_u\right) \Xi\left(\boldsymbol{\kappa}_u\right) \, d\mu_u, \tag{1.8}$$

and hence an even solution to (1.7) restricted to $\mathscr{C}^n_{R,d}$ satisfies (1.5) with the correct boundary conditions. It is also clear that a solution to (1.5) with Neumann boundary condition will extend evenly to a solution of (1.7), compare Figures 1.1 and 1.2.

As before we have the specific case of the volume preserving mean curvature flow:

$$\frac{\partial u}{\partial t} = \sqrt{1 + \left|\tilde{\nabla} u\right|^2} \left(\fint_{\mathscr{T}^n_{R,d}} H\left(u\right) \, d\mu_u - H\left(u\right) \right), \qquad u\left(\cdot, 0\right) = u_0. \tag{1.9}$$

1.2 Overview

The remainder of this thesis is split into seven chapters. Chapter 2 provides some background to the differential geometry used in this thesis. We start by including some definitions of important quantities of a hypersurface and useful curvature identities. The evolution of these quantities under a flow of the form (1.4), along with formulas for how to calculate them in the case of a normal graph are also given. The necessary functional analysis background is provided in Chapter 3, including definitions of

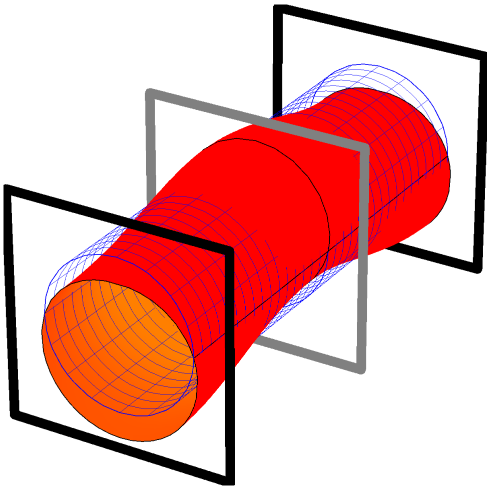


Figure 1.2: An extension of a graph over a cylinder satisfying the Neumann boundary condition (Figure 1.1) to a graph over the torus

interpolation spaces and sectorial operators. The experienced reader may skip these chapters.

In Chapter 4 we prove two short time existence results for the weighted volume preserving curvature flows and an improved version for the volume preserving mean curvature flow case. We start by calculating the linearisation of the speed operator about both the sphere and cylinder and then extend it to give the linearisation of the speed in (1.7). The linearised operators are then shown to be sectorial, which is the main assumption needed in order to obtain short time existence of the flows. These theorems are both local in nature, as they only apply to hypersurfaces that have a height function that is small in a little-Hölder space. The chapter finishes by improving on this for the volume preserving mean curvature flow of graphs over cylinders. In this case short time existence is proved for all valid height functions; further, the solution is found to be smooth after the initial time.

The question of stability of spheres under the weighted volume preserving curvature flows is addressed in Chapter 5. Through calculation of the eigenvalues of the linearised speed operator the sphere is found to be linearly stable (all eigenvalues are non-positive). A locally invariant exponentially attractive center manifold is found to exist for the flow and it is proven to consist entirely of functions whose graph is a sphere. Thus we obtain the stability result: if the initial graph function is small, then under (1.5) it will converge exponentially fast to a function whose graph is a sphere. Chapter 6 covers similar material for the case of graphs over a cylinder by analysing (1.7). However, in this case the eigenfunctions in the flat direction can yield positive eigenvalues. For the system to be linearly stable we require the assumption:

$$R > \frac{d}{\pi}\sqrt{\frac{\frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right)(n-1)}{\frac{\partial F}{\partial \kappa_n}\left(\boldsymbol{\kappa}_0\right)}}.$$

Chapter 7 is split into three sections. The first deals with improving the results of Chapter 6 in the case of volume preserving mean curvature flow. It uses the short time existence result proved at the end of Chapter 4 and a bootstrapping method to obtain convergence with respect to stricter norms. Section two investigates the condition on the radius of the cylinder, i.e. that cylinders are only stable if $R > \frac{d\sqrt{n-1}}{\pi}$. To show that this condition is necessary to obtain stability, the simplified case of axially symmetric flow is considered. By introducing a parameter, that depends solely on the enclosed volume of the hypersurface, the flow is shown to be equivalent to a PDE on the space of average zero function . A bifurcation analysis of the stationary solutions to this PDE is undertaken. It is found that there is a continuously differentiable curve of non-cylindrical stationary solutions that passes through a cylinder of radius $R = \frac{d\sqrt{n-1}}{\pi}$. This means that any open neighbourhood about a cylinder of this radius must contain a non-cylindrical stationary solution to the flow. Further analysis shows that the stationary solutions on the curve close to the cylinder are unstable in dimensions ten and under but are stable under axially symmetric volume preserving perturbations in dimensions eleven and above. The last section of this chapter deals with determining the height functions for these stationary solutions explicitly. The volume enclosed by the hypersurfaces is also calculated and the bifurcation curve plotted, in order to highlight the change in stability as the dimension increases.

Lastly, in Chapter 8 we investigate the classical MCF and show how the techniques in this paper can be applied to the MCF setting. As an example we consider normal graphs over catenoids. The speed operator linearised about zero is found to be a sectorial operator and we obtain a local short time existence result. A spectral analysis of the operator shows the catenoid is linearly unstable and we prove the existence of stable and unstable manifolds for the flow.

1.3 Notation and Definitions

In this section we define some of the notation and conventions that will be used throughout the thesis. We will use the Latin characters $i, j, k, \ldots$ as indices and we use the Einstein summation convention to sum over repeated indices, unless explicitly stated. In the cases that we do not employ the Einstein summation convention we use the indices $a, b, \ldots$. The Kronecker delta will be denoted δ^i_j, which is equal to one if $i = j$ and zero otherwise.

When dealing with normal graphs we will use the notation $\mathring{g}_{kl}$ and $\mathring{h}^k_i$ to refer to the metric and Weingarten map of the base hypersurface, Ω_0. Often we will need to consider the inverse of the tensor $\left(\delta^k_i + \rho\mathring{h}^k_i\right)\mathring{g}_{kl}\left(\delta^l_j + \rho\mathring{h}^l_j\right)$, so we define this to be $(\tilde{g}_\rho)^{ij}$ and also define $\left|\tilde{\nabla}\rho\right|^2 := (\tilde{g}_\rho)^{ij}\nabla_i\rho\nabla_j\rho$, where ∇ is the Levi-Civita connection on Ω_0.

We use the notation $\mathscr{S}^n_R$ to represent a sphere of radius R and $\mathscr{C}^n_{R,d} = \mathscr{S}^{n-1}_R \times (0, d)$ to represent a cylinder of radius R and length d. The torus will be denoted by $\mathscr{T}^n_{R,d} = \mathscr{S}^{n-1}_R \times \mathscr{S}^1_{\frac{d}{\pi}}$, and it will be equipped with the 'flat' metric obtained by evenly extending the $\mathscr{C}^n_{R,d}$ metric. We consider the local coordinates on the cylinder and torus given by $\boldsymbol{p} = (\boldsymbol{q}, z)$, with $\boldsymbol{q}$ a point on the sphere (in local coordinates), $0 < z < d$ for the cylinder and $-d < z \leq d$ for the torus.

Throughout the thesis f, v will be used to denote general functions on a manifold, while a function on $\mathscr{S}^n_R$ or $\mathscr{C}^n_{R,d}$ will be denoted by ρ and a function on $\mathscr{T}^n_{R,d}$ will be denoted by u. We will often need to move between a bounded, continuous function on the cylinder with boundary and a function on the torus, hence we make use of the notation:

$$u_\rho = u_\rho(\boldsymbol{q}, z) := \begin{cases} \rho(\boldsymbol{q}, z) & z \in [0, d], \\ \rho(\boldsymbol{q}, -z) & z \in (-d, 0). \end{cases} \tag{1.10}$$

Likewise when moving from a function on a torus to a function on the cylinder we define the restriction:

$$u|_{\overline{\mathscr{C}}^n_{R,d}} = u|_{\overline{\mathscr{C}}^n_{R,d}}(\boldsymbol{q}, z) := u(\boldsymbol{q}, z) \quad z \in [0, d], \boldsymbol{q} \in \mathscr{S}^{n-1}_R, \tag{1.11}$$

in the case $n = 1$ we use the notation $u|_{[0,d]}$.

The characters X, Y, Z will often be used to denote Banach spaces. An open ball in a space X of radius r centred at a point x will be denoted $B_{X,r}(x)$. When we consider a function space on a manifold with boundary, $X\left(\bar{M}^n\right)$, we often have functions that satisfy the boundary condition, $B\left[f\right]|_{\partial M} = 0$; we therefore define:

$$X_B\left(\bar{M}^n\right) := \left\{f \in X\left(\bar{M}^n\right) : B\left[f\right]|_{\partial M^n} = 0\right\}.$$

The characters O, U, V, W will be used to denote open sets. In particular we define the subspaces of valid graph functions over the sphere, torus and cylinder:

$$U_{k,\alpha} := \left\{\rho \in h^{k,\alpha}\left(\mathscr{S}^n_R\right) : \rho > -R\right\}, \tag{1.12}$$

$$V_{k,\alpha} := \left\{u \in h^{k,\alpha}\left(\mathscr{T}^n_{R,d}\right) : u > -R\right\}, \tag{1.13}$$

$$\tilde{V}_{k,\alpha} := \left\{\rho \in h^{k,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right) : \rho > -R\right\}, \tag{1.14}$$

see Section 3.1 for a definition of the little-Hölder spaces $h^{k,\alpha}$.

For a nonlinear operator $G : Y \to X$ we denote the Fréchet derivative of G by ∂G. In the case where G has multiple arguments we use a subscript, e.g. ∂_2, to indicate which argument it is with respect to. The space of linear operators from Y to X will be denoted $\mathcal{L}\left(Y, X\right)$ and for a linear operator $A : Y \subset X \to X$ we denote its spectrum by $\sigma\left(A\right)$ and resolvent set by $\rho\left(A\right)$. We also define the following subsets of the spectrum:

$$\sigma_+\left(A\right) := \left\{\lambda \in \sigma\left(A\right) : Re\left(\lambda\right) \geq 0\right\}, \tag{1.15}$$

$$\sigma_-\left(A\right) := \left\{\lambda \in \sigma\left(A\right) : Re\left(\lambda\right) < 0\right\}, \tag{1.16}$$

$$\sigma_>\left(A\right) := \left\{\lambda \in \sigma\left(A\right) : Re\left(\lambda\right) > 0\right\}, \tag{1.17}$$

and the spectral constants:

$$\omega_- := -\sup_{\lambda \in \sigma_-(A)} Re(\lambda), \tag{1.18}$$

$$\omega_+ := \inf_{\lambda \in \sigma_>(A)} Re(\lambda). \tag{1.19}$$

In the case where $\sigma_+\left(A\right)$ consists of a finite number of isolated eigenvalues, we denote its spectral projection by P_+. That is, $P_+ : X \to X$ such that if we define $A_+ := A|_{P_+(Y)}$ and $A_- := A|_{(I-P_+)(Y)}$, then $\sigma\left(A_+\right) = \sigma_+\left(A\right)$ and $\sigma\left(A_-\right) = \sigma_-\left(A\right)$.

2

Differential Geometry Background

This chapter is designed to give an overview of the differential geometry knowledge used within the thesis. We start by investigating the properties of an immersed hypersurface. Section 2.2 discusses the flows that will be considered throughout the thesis. In Section 2.3 we will consider the specific case of normal graphs and recast the geometric quantities in terms of the graph function.

2.1 Hypersurfaces

Consider an n-dimensional manifold M^n, an immersion $\boldsymbol{X} : M^n \to \mathbb{R}^{n+1}$ and let $\Omega \subset \mathbb{R}^{n+1}$ be the image of M^n under this immersion. Local coordinates on M^n will be denoted by $x^1, \ldots, x^n$ and, by using "$\cdot$" to denote the inner product on $\mathbb{R}^{n+1}$, the metric, g, of Ω induced by the immersion $\boldsymbol{X}$ is given in component form by:

$$g_{ij} = \frac{\partial \boldsymbol{X}}{\partial x^i} \cdot \frac{\partial \boldsymbol{X}}{\partial x^j}.$$

The components of the inverse metric, g^{-1}, will be denoted g^{ij}. The normal to Ω is denoted by $\boldsymbol{\nu}$ and the second fundamental form, $A = (h_{ij})$, can be calculated from:

$$h_{ij} = -\frac{\partial^2 \boldsymbol{X}}{\partial x^i \partial x^j} \cdot \boldsymbol{\nu}.$$

The Weingarten map can then be represented by the matrix $\mathscr{W} = \left(h_i^j\right) = \left(g^{ik} h_{kj}\right)$, the eigenvalues of this matrix are the principal curvatures of Ω and are denoted by

κ_a. Other important curvature terms include the norm of the second fundamental form $|A| = \left(g^{ij}g^{kl}h_{ik}h_{jl}\right)^{1/2}$ and the elementary symmetric functions of the principal curvatures:

$$E_a := \begin{cases} 1 & a = 0, \\ \sum_{1\leq b_1 < \ldots < b_a \leq n} \prod_{i=1}^{a} \kappa_{b_i} & a = 1, \ldots, n, \end{cases} \tag{2.1}$$

note that $E_1 = H$, the mean curvature.

When taking derivatives of tensor fields on M^n we will often use the Levi-Civita connection ∇, which for a (r,s)-tensor T is given by:

$$\nabla_k T^{j_1\ldots j_r}_{i_1\ldots i_s} = \frac{\partial T^{j_1\ldots j_r}_{i_1\ldots i_s}}{\partial x^k} + \Gamma^{j_1}_{kl} T^{l j_2\ldots j_r}_{i_1\ldots i_s} + \ldots + \Gamma^{j_r}_{kl} T^{j_1\ldots j_{r-1} l}_{i_1\ldots i_s} - \Gamma^l_{k i_1} T^{j_1\ldots j_r}_{l i_2\ldots i_s} - \ldots - \Gamma^l_{k i_s} T^{j_1\ldots j_r}_{i_1\ldots i_{s-1} l},$$

where Γ^k_{ij} are the Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

The hypersurface Laplacian will be denoted by $\Delta := g^{ij}\nabla_i\nabla_j$ and the hypersurface measure by $d\mu = \sqrt{\det(g)}\, d\boldsymbol{x}$. For a compact hypersurface Ω we also have the quantities:

$$V_a := \begin{cases} \left((n+1)\binom{n}{a}\right)^{-1} \int_{M^n} E_{n-a}\, d\mu & a = 0, \ldots, n \\ Vol(\Phi) & a = n+1, \end{cases} \tag{2.2}$$

where Φ is an $(n+1)$-dimensional region associated to Ω. For a closed hypersurface Φ is the enclosed volume, while if the hypersurface is a graph over a cylinder the volume is that enclosed by the hypersurface and the end hyperplanes. The area of the hypersurface, $|\Omega|$, is proportional to V_n, i.e. $|\Omega| = (n+1)V_n$, and in the case where Ω is convex V_a coincides with the a^{th} mixed volume, see [5] for a definition. The average of a function $f : M^n \to \mathbb{R}$ is denoted by:

$$⨍_{M^n} f\, d\mu := \frac{1}{|\Omega|} \int_{M^n} f\, d\mu.$$

Various important identities involve the second fundamental form; we provide some here that are used in the study of curvature flows. The Codazzi equations state that ∇A is a fully symmetric $(0,3)$-tensor:

$$\nabla_k h_{ij} = \nabla_i h_{jk} = \nabla_j h_{ki}. \tag{2.3}$$

The Gauss-Weingarten relations use the tangent vectors and normal of Ω as a basis for $\mathbb{R}^{n+1}$ in order to express the second derivative of the immersion:

$$\frac{\partial^2 \boldsymbol{X}}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial \boldsymbol{X}}{\partial x^k} - h_{ij}\boldsymbol{\nu}, \tag{2.4}$$

and the derivative of the normal:

$$\frac{\partial \boldsymbol{\nu}}{\partial x^i} = h_i^k \frac{\partial \boldsymbol{X}}{\partial x^k}. \tag{2.5}$$

Using the formula for the Levi-Civita connection, (2.4) can also be expressed as:

$$\nabla_i \nabla_j \boldsymbol{X} = -h_{ij} \boldsymbol{\nu}.$$

Lastly we have that the elementary symmetric functions of the principal curvatures satisfy the identity, found in Equation (5.86) of [24] also see Appendix B for a complete proof:

$$\frac{\partial E_{a+1}}{\partial h_j^i} = E_a \delta_i^j - h_m^j \frac{\partial E_a}{\partial h_m^i}, \tag{2.6}$$

where $a = 0, \dots, n$ (in the $a = n$ case we use the convention $E_{n+1} = 0$).

2.2 Curvature Flows

We present here some evolution equations for the properties of a family of hypersurfaces undergoing a flow of the form in equation (1.4), a derivation can be found in [4]. For ease we do not show the explicit dependence on $\boldsymbol{\kappa}$ in this section.

Lemma 2.2.1.

(a)

$$\frac{\partial g_{ij}}{\partial t} = 2\left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) h_{ij}$$

(b)

$$\frac{\partial \, d\mu}{\partial t} = \left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) H \, d\mu$$

(c)

$$\frac{\partial \boldsymbol{\nu}}{\partial t} = g^{ij} \nabla_i F \frac{\partial \boldsymbol{X}}{\partial x^j}$$

(d)

$$\frac{\partial h_j^j}{\partial t} = g^{im} \nabla_m \nabla_j F - \left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) h_m^i h_j^m$$

From these equations we are able to calculate how the quantities V_a evolve under (1.4) and find that mixed volume preserving flows, i.e. when $\Xi = E_{a+1}$, preserve V_{n-a}.

Lemma 2.2.2.

$$\frac{dV_a}{dt} = \begin{cases} 0 & a = 0, \\ \binom{n+1}{a}^{-1} \int_M E_{n+1-a} \left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F\Xi \, d\mu - F \right) d\mu & a = 1, \ldots, n+1. \end{cases}$$

Proof. The case of $l = n+1$ follows immediately from the first variation of volume, which gives:

$$\frac{dV_{n+1}}{dt} = \int_{M^n} \frac{\partial \boldsymbol{X}}{\partial t} \cdot \boldsymbol{\nu} \, d\mu.$$

The other cases are given by McCoy in Lemma 4.3 of [40] for the case where Ω_t are convex hypersurfaces. McCoy uses the definition of mixed volumes of convex hypersurfaces, see [5], which are not valid unless the hypersurface is convex. To obtain the result for all solutions to the flow we take the divergence the identity in (2.6) to obtain for $a = 0, \ldots, n$:

$$\begin{aligned} g^{ki} \nabla_k \left(\frac{\partial E_{a+1}}{\partial h^i_j} \right) =& g^{kj} \nabla_k E_a - g^{ki} \nabla_k h^j_m \frac{\partial E_a}{\partial h^i_m} - g^{ki} h^j_m \nabla_k \left(\frac{\partial E_a}{\partial h^i_m} \right) \\ =& g^{kj} \nabla_k h^i_m \frac{\partial E_a}{\partial h^i_m} - g^{ki} g^{lj} \nabla_k h_{lm} \frac{\partial E_a}{\partial h^i_m} - h^j_m g^{ki} \nabla_k \left(\frac{\partial E_a}{\partial h^i_m} \right) \\ =& g^{kj} \nabla_k h^i_m \frac{\partial E_a}{\partial h^i_m} - g^{ki} g^{lj} \nabla_l h_{km} \frac{\partial E_a}{\partial h^i_m} - h^j_m g^{ki} \nabla_k \left(\frac{\partial E_a}{\partial h^i_m} \right) \\ =& - h^j_m g^{ki} \nabla_k \left(\frac{\partial E_a}{\partial h^i_m} \right), \end{aligned}$$

where to get to the second last line we use the Codazzi equation (2.3). Hence, due to $g^{ki} \nabla_k \left(\frac{\partial E_0}{\partial h^i_j} \right) = 0$ we have that $g^{ki} \nabla_k \left(\frac{\partial E_a}{\partial h^i_j} \right) = 0$ for all $a = 0, \ldots, n$. Now we can derive the evolution equation:

$$\begin{aligned} (n+1) \binom{n}{a} \frac{dV_a}{dt} =& \int_{M^n} \frac{\partial E_{n-a}}{\partial t} + \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) H E_{n-a} \, d\mu \\ =& \int_{M^n} \frac{\partial E_{n-a}}{\partial h^i_j} \frac{\partial h^i_j}{\partial t} + \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) H E_{n-a} \, d\mu \\ =& \int_{M^n} \frac{\partial E_{n-a}}{\partial h^i_j} g^{im} \nabla_m \nabla_j F - \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) \frac{\partial E_{n-a}}{\partial h^i_j} h^i_m h^m_j \\ & + \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) H E_{n-a} \, d\mu \end{aligned}$$

$$
\begin{aligned}
(n+1)\binom{n}{a}\frac{dV_a}{dt} =& \int_{M^n} \nabla_m \left(\frac{\partial E_{n-a}}{\partial h^i_j} g^{im} \nabla_j F \right) + \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) H E_{n-a} \\
&+ \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) h^i_m \left(\frac{\partial E_{n+1-a}}{\partial h^i_m} - E_{n-a}\delta^m_i \right) d\mu \\
=&(n+1-a) \int_{M^n} \left(\frac{\int_{M^n} F\Xi \, d\mu}{\int_{M^n} \Xi \, d\mu} - F \right) E_{n+1-a} \, d\mu,
\end{aligned}
$$

where the second last line is due to (2.6) and the last line is due to the homogeneity of E_{n+1-a}. $\square$

Corollary 2.2.3. *If $\Xi = E_{a+1}$ then (1.4) is the mixed volume preserving curvature flow and it preserves V_{n-a} as long as the flow exists.*

2.3 Normal Graphs

Consider an embedding of a hypersurface $\boldsymbol{X}_0 : M^n \to \mathbb{R}^{n+1}$, which has metric $\mathring{g}$, second fundamental form $\mathring{A} = \left(\mathring{h}_{ij}\right)$, Weingarten map $\mathring{W}$ and normal $\boldsymbol{\nu}_0$. Let Ω_ρ be a normal graph over $\Omega_0 := \boldsymbol{X}_0\left(M^n\right)$ given by the height function $\rho : M^n \to \mathbb{R}$. Ω_ρ can be represented by the embedding:

$$\boldsymbol{X}_\rho = \boldsymbol{X}_0 + \rho \boldsymbol{\nu}_0 \tag{2.7}$$

Such a graph hypersurface is well defined for any ρ that satisfies $\|\rho\|_{L^\infty} < \frac{1}{\kappa_{max}}$, where $\kappa_{max} = \max_{a\in[1,n]} \|\kappa_a\|_{L^\infty}$ is the maximum of the absolute values of the principal curvatures of Ω_0. Note that when $M^n = \mathscr{S}^n_R$ or $\mathscr{C}^n_{R,d}$ we will always take $\boldsymbol{X}_0$ to be the natural embedding.

Lemma 2.3.1. *The tangent vectors, metric, inverse metric and normal of the hypersurface Ω_ρ are given by:*

$$\frac{\partial \boldsymbol{X}_\rho}{\partial x^i} = \left(\delta^k_i + \rho \mathring{h}^k_i\right) \frac{\partial \boldsymbol{X}_0}{\partial x^k} + \nabla_i \rho \boldsymbol{\nu}_0, \tag{2.8}$$

$$(g_\rho)_{ij} = \left(\delta^k_i + \rho \mathring{h}^k_i\right) \mathring{g}_{kl} \left(\delta^l_j + \rho \mathring{h}^l_j\right) + \nabla_i \rho \nabla_j \rho, \tag{2.9}$$

$$(g_\rho)^{ij} = (\tilde{g}_\rho)^{ij} - \left(1 + \left|\tilde{\nabla}\rho\right|^2\right)^{-1} (\tilde{g}_\rho)^{ik} (\tilde{g}_\rho)^{jl} \nabla_k \rho \nabla_l \rho, \tag{2.10}$$

and

$$\boldsymbol{\nu}_\rho = \frac{1}{\sqrt{1 + \left|\tilde{\nabla}\rho\right|^2}} \left(\boldsymbol{\nu}_0 - (\tilde{g}_\rho)^{rp} \left(\delta^s_p + \rho \mathring{h}^s_p\right) \nabla_r \rho \frac{\partial \boldsymbol{X}_0}{\partial x^s} \right), \tag{2.11}$$

where $(\tilde{g}_\rho)^{ij}$ *is the inverse of* $\left(\delta_i^k + \rho \mathring{h}_i^k\right) \mathring{g}_{kl} \left(\delta_j^l + \rho \mathring{h}_j^l\right)$ *and* $\left|\tilde{\nabla}\rho\right|^2 = (\tilde{g}_\rho)^{ij} \nabla_i \rho \nabla_j \rho$.

Note that due to the restriction on the size of the height function the quantity $\left(I + \rho \mathring{W}\right)$ is always invertible.

Proof. The first equation follows directly from the Gauss Weingarten relation in (2.5) and the second equation follows from the definition of the induced metric. The formula for the inverse metric is given in terms of $(\tilde{g}_\rho)^{ij}$, which is itself defined as the inverse of a $(0,2)$-tensor. While this may seem unusual, we will see later that the quantity $(\tilde{g}_\rho)^{ij}$ has an alternate interpretation that makes its calculation simpler.

To calculate the equation of the normal we let $\boldsymbol{\nu}_\rho = b(\rho)\left(\boldsymbol{\nu}_0 + c(\rho)^l \frac{\partial \boldsymbol{X}_0}{\partial x^l}\right)$ and dot it with the tangent vectors:

$$0 = b(\rho)\left(\nabla_i \rho + c(\rho)^l \left(\delta_i^k + \rho \mathring{h}_i^k\right) \mathring{g}_{kl}\right).$$

Therefore:

$$c(\rho)^l = -\mathring{g}^{kl}\left(I + \rho \mathring{W}\right)^{-1}{}_k^i \nabla_i \rho = -(\tilde{g}_\rho)^{lk}\left(\delta_k^i + \rho \mathring{h}_k^i\right) \nabla_i \rho.$$

The quantity $b(\rho)$ is found using the unit vector condition and noting that for the direction of $\boldsymbol{\nu}_\rho$ and $\boldsymbol{\nu}_0$ to be consistent, $b(\rho)$ is positive.

$$\begin{aligned}
b(\rho)^{-2} &= 1 + (\tilde{g}_\rho)^{rp}\left(\delta_p^s + \rho (h_\rho)_p^s\right)(\tilde{g}_\rho)^{iq}\left(\delta_q^j + \rho \mathring{h}_q^j\right) \nabla_r \rho \nabla_i \rho \mathring{g}_{sj} \\
&= 1 + \delta_q^r (\tilde{g}_\rho)^{iq} \nabla_r \rho \nabla_i \rho \\
&= 1 + \left|\tilde{\nabla}\rho\right|^2.
\end{aligned}$$

□

We note here that if we consider the foliation of normal graphs given by constant ρ then $(\tilde{g}_\rho)^{ij}$ is the inverse metric of the corresponding foliation; this simplifies calculations for many hypersurfaces. For example if $M^n = \mathscr{S}_R^n$ then $(\tilde{g}_\rho)^{ij} = \frac{R^2}{(R+\rho)^2}\mathring{g}^{ij}$. We now consider some curvature quantities for Ω_ρ and define $L(\rho) := \sqrt{1 + \left|\tilde{\nabla}\rho\right|^2}$.

Lemma 2.3.2. *The second fundamental form and mean curvature of* Ω_ρ *are given by:*

$$\begin{aligned}
(h_\rho)_{ij} =& L(\rho)^{-1}\Big(-\nabla_i \nabla_j \rho + (\delta_i^s + \rho \mathring{h}_i^s)\mathring{h}_{sj} + (\tilde{g}_\rho)^{rp}(\delta_p^s + \rho \mathring{h}_p^s)\nabla_r \rho \left(\mathring{h}_{is}\nabla_j \rho + \mathring{h}_{js}\nabla_i \rho\right) \\
&+ \rho (\tilde{g}_\rho)^{rp}(\delta_p^s + \rho \mathring{h}_p^s)\nabla_s \mathring{h}_{ij}\nabla_r \rho\Big) \qquad (2.12)
\end{aligned}$$

and

$$\begin{aligned}H(\rho) =&L(\rho)^{-3}(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\left(\delta_i^s+\rho\mathring{h}_i^s\right)\mathring{h}_{sj}\nabla_k\rho\nabla_l\rho+L(\rho)^{-1}(\tilde{g}_\rho)^{ij}\left(\delta_i^s+\rho\mathring{h}_i^s\right)\mathring{h}_{sj}\\&+\rho L(\rho)^{-3}\left(L(\rho)^2(\tilde{g}_\rho)^{ij}-(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\nabla_k\rho\nabla_l\rho\right)(\tilde{g}_\rho)^{pr}\left(\delta_p^s+\rho\mathring{h}_p^s\right)\nabla_s\mathring{h}_{ij}\nabla_r\rho\\&-L(\rho)^{-3}\left(L(\rho)^2(\tilde{g}_\rho)^{ij}-(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\nabla_k\rho\nabla_l\rho\right)\nabla_i\nabla_j\rho. \end{aligned}\tag{2.13}$$

Proof. We start by calculating the second derivative of the embedding using the Gauss-Weingarten relations (2.4) and (2.5):

$$\begin{aligned}\frac{\partial^2\boldsymbol{X}_\rho}{\partial x^i\partial x^j}&=\left(\mathring{h}_i^l\nabla_j\rho+\rho\frac{\partial\mathring{h}_i^l}{\partial x^j}\right)\frac{\partial\boldsymbol{X}_0}{\partial x^l}+\left(\delta_i^l+\rho\mathring{h}_i^l\right)\frac{\partial^2\boldsymbol{X}_0}{\partial x^l\partial x^j}+\frac{\partial^2\rho}{\partial x^i\partial x^j}\boldsymbol{\nu}_0+\nabla_i\rho\frac{\partial\boldsymbol{\nu}_0}{\partial x^j}\\&=\left(\mathring{h}_i^k\nabla_j\rho+\rho\frac{\partial\mathring{h}_i^k}{\partial x^j}+\mathring{\Gamma}_{ij}^k+\rho\mathring{h}_i^l\mathring{\Gamma}_{lj}^k+\mathring{h}_j^k\nabla_i\rho\right)\frac{\partial\boldsymbol{X}_0}{\partial x^k}\\&\quad+\left(\frac{\partial^2\rho}{\partial x^i\partial x^j}-\mathring{h}_{lj}\left(\delta_i^l+\rho\mathring{h}_i^l\right)\right)\boldsymbol{\nu}_0\\&=\left(\mathring{h}_i^k\nabla_j\rho+\mathring{h}_j^k\nabla_i\rho+\rho\nabla_j\mathring{h}_i^k+\left(\delta_l^k+\rho\mathring{h}_l^k\right)\mathring{\Gamma}_{ij}^l\right)\frac{\partial\boldsymbol{X}_0}{\partial x^k}\\&\quad+\left(\frac{\partial^2\rho}{\partial x^i\partial x^j}-\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)\right)\boldsymbol{\nu}_0\end{aligned}$$

The last line used the equation for the covariant derivative of the Weingarten map, $\nabla_j\mathring{h}_i^k=\frac{\partial\mathring{h}_i^k}{\partial x^j}+\mathring{\Gamma}_{lj}^k\mathring{h}_i^l-\mathring{\Gamma}_{ij}^l\mathring{h}_l^k$. Using the definition of the second fundamental form and equation (2.11) we obtain:

$$\begin{aligned}(h_\rho)_{ij}=&L(\rho)^{-1}\left(\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)-\frac{\partial^2\rho}{\partial x^i\partial x^j}+(\tilde{g}_\rho)^{rp}(\delta_p^s+\rho\mathring{h}_p^s)\nabla_r\rho\left(\mathring{h}_{is}\nabla_j\rho+\mathring{h}_{js}\nabla_i\rho\right)\right.\\&\left.+\mathring{\Gamma}_{ij}^r\nabla_r\rho+\rho(\tilde{g}_\rho)^{rp}(\delta_p^s+\rho\mathring{h}_p^s)\nabla_r\rho\nabla_j\mathring{h}_{is}\right),\end{aligned}$$

which gives (2.12) by converting the partial derivatives of ρ to covariant derivatives and using the Codazzi equation (2.3). Equation (2.13) follows from (2.10), (2.12) and the definition of $H(\rho)=(g_\rho)^{ij}(h_\rho)_{ij}$, note that:

$$\begin{aligned}&(g_\rho)^{ij}\left(\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)+(\tilde{g}_\rho)^{rp}(\delta_p^s+\rho\mathring{h}_p^s)\nabla_r\rho\left(\mathring{h}_{is}\nabla_j\rho+\mathring{h}_{js}\nabla_i\rho\right)\right)\\&\quad=2\left((\tilde{g}_\rho)^{ij}-L(\rho)^{-2}(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\nabla_k\rho\nabla_l\rho\right)(\tilde{g}_\rho)^{rp}(\delta_p^s+\rho\mathring{h}_p^s)\mathring{h}_{is}\nabla_r\rho\nabla_j\rho\\&\qquad+(\tilde{g}_\rho)^{ij}\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)-L(\rho)^{-2}(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)\nabla_k\rho\nabla_l\rho\\&\quad=2(\tilde{g}_\rho)^{ij}(\tilde{g}_\rho)^{rp}(\delta_p^s+\rho\mathring{h}_p^s)\mathring{h}_{is}\nabla_r\rho\nabla_j\rho-L(\rho)^{-2}(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)\nabla_k\rho\nabla_l\rho\\&\qquad+(\tilde{g}_\rho)^{ij}\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)-2L(\rho)^{-2}(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{rp}\left|\tilde{\nabla}\rho\right|^2(\delta_p^s+\rho\mathring{h}_p^s)\mathring{h}_{is}\nabla_k\rho\nabla_r\rho\\&\quad=(\tilde{g}_\rho)^{ij}\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)+L(\rho)^{-2}(\tilde{g}_\rho)^{ik}(\tilde{g}_\rho)^{jl}\mathring{h}_{sj}\left(\delta_i^s+\rho\mathring{h}_i^s\right)\nabla_k\rho\nabla_l\rho.\end{aligned}$$

□

We finish this chapter by proving that a solution to equation (1.5) is equivalent to a solution to (1.4).

Lemma 2.3.3. *Let $\rho : M^n \times [0,T)$ be a solution to (1.5) with initial condition ρ_0, then $\boldsymbol{X}_\rho$ is tangentially diffeomorphic to the solution of (1.4) with initial condition $\boldsymbol{X}_{\rho_0}$.*

Proof. Let $\phi : M^n \times [0,T) \to M^n$ be a diffeomorphism satisfying the system:

$$\frac{\partial \phi^i}{\partial t} = -\left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) \frac{(\tilde{g}_\rho)^{ij} \nabla_j \rho}{\sqrt{1 + \left|\tilde{\nabla}\rho\right|^2}},$$

and set $\tilde{\boldsymbol{X}}(\boldsymbol{p}, t) = \boldsymbol{X}_\rho(\phi(\boldsymbol{p}, t), t)$. Then, using equations (2.8) and (2.11), $\tilde{\boldsymbol{X}}$ satisfies (1.4):

$$\begin{aligned}
\frac{\partial \tilde{\boldsymbol{X}}}{\partial t} =& \frac{\partial \boldsymbol{X}_\rho}{\partial t} + \frac{\partial \boldsymbol{X}_\rho}{\partial x^i} \frac{\partial \phi^i}{\partial t} \\
=& \frac{\partial \rho}{\partial t} \boldsymbol{\nu}_0 - \left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) \frac{(\tilde{g}_\rho)^{ij} \nabla_j \rho}{\sqrt{1 + \left|\tilde{\nabla}\rho\right|^2}} \left(\left(\delta_i^k + \rho \mathring{h}_i^k\right) \frac{\partial \boldsymbol{X}_0}{\partial x^k} + \nabla_i \rho \boldsymbol{\nu}_0\right) \\
=& \frac{1}{\sqrt{1 + \left|\tilde{\nabla}\rho\right|^2}} \left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) \left(\boldsymbol{\nu}_0 - (\tilde{g}_\rho)^{ij} \nabla_j \left(\delta_i^k + \rho \mathring{h}_i^k\right) \frac{\partial \boldsymbol{X}_0}{\partial x^k}\right) \\
=& \left(\frac{1}{\int_{M^n} \Xi \, d\mu} \int_{M^n} F \Xi \, d\mu - F\right) \boldsymbol{\nu}_\rho.
\end{aligned}$$

□

3

Functional Analysis Background

This chapter is designed to give an overview of the functional analysis knowledge used within the thesis. We will introduce interpolation spaces for a Banach couple and define the little-Hölder spaces, which are their own interpolation spaces. In Section 3.2 we will define what it means for an operator to be sectorial, as well as prove that an elliptic operator on the little-Hölder spaces is sectorial. The section ends with some results for perturbations of sectorial operators.

3.1 Interpolation Spaces

The continuous interpolation spaces that we consider in this thesis are defined for a Banach couple $Z \subset Y$ and are given by the interpolation functor $(Y, Z)_\theta$, where $\theta \in (0,1)$. They are defined, see [38], as follows:

$$(Y, Z)_\theta := \left\{ f \in Y : \lim_{t \to 0^+} t^{-\theta} K\left(t, f, Y, Z\right) = 0 \right\},$$

where

$$K\left(t, f, Y, Z\right) := \inf_{g \in Z} \left(\|f - g\|_Y + t \|g\|_Z \right).$$

The norms on these spaces are:

$$\|f\|_{(Y,Z)_\theta} := \left\| t^{-\theta} K\left(t, f, Y, Z\right) \right\|_{L^\infty(0,\infty)}.$$

The reiteration theorem for interpolation spaces allows for easier characterisation of interpolation spaces.

Theorem 3.1.1 (Remark 1.2.16 [38]). *For $\theta_0, \theta_1, \theta_2 \in (0,1)$ and Y, Z Banach spaces such that $Z \subset Y$:*

$$\left((Y,Z)_{\theta_1}, Z\right)_{\theta_0} = (Y,Z)_{(1-\theta_0)\theta_1+\theta_0}, \quad \left(Y, (Y,Z)_{\theta_1}\right)_{\theta_0} = (Y,Z)_{\theta_1\theta_0}.$$

An immediate consequence is:

$$\left((Y,Z)_{\theta_1}, (Y,Z)_{\theta_2}\right)_{\theta_0} = (Y,Z)_{(1-\theta_0)\theta_1+\theta_0\theta_2}. \tag{3.1}$$

Another useful result relates to interpolating between Z and a closed subspace of Y.

Lemma 3.1.2. *Let Y, Z be Banach spaces such that $Z \subset Y$ and U be a closed subspace of Y that has an associated projection $P : Y \to U$ with the properties:*

$$\|P[y]\|_Y \le C_1 \|y\|_Y, \text{ for all } y \in Y \text{ and } \|P[z]\|_Z \le C_2 \|z\|_Z, \text{ for all } z \in Z. \tag{3.2}$$

Then

$$(Y,Z)_\theta \cap U = (U, Z \cap U)_\theta, \text{ for all } \theta \in (0,1), \tag{3.3}$$

where U is endowed with the same norm as Y and $Z \cap U$ has the same norm as Z.

Proof. We fix $\theta \in (0,1)$ and suppose that $x \in (U, Z \cap U)_\theta$. Since U and Y have the same norm we have that $K(t,x,U,Z\cap U) = K(t,x,Y,Z\cap U)$ for all $t > 0$ and by taking the infinum over the larger space we therefore have $K(t,x,U,Z\cap U) \ge K(t,x,Y,Z)$ for all $t > 0$. Therefore

$$0 = \lim_{t\to 0^+} t^{-\theta} K(t,x,U,Z\cap U) \ge \lim_{t\to 0^+} t^{-\theta} K(t,x,Y,Z) \ge 0,$$

and hence $x \in (Y,Z)_\theta \cap U$, so $(U, Z\cap U)_\theta \subset (Y,Z)_\theta \cap U$.

Now suppose $x \in (Y,Z)_\theta \cap U$ then for all $t > 0$ and $z \in Z$ we can use (3.2) to obtain the estimate

$$\|x - P[z]\|_Y + t\|P[z]\|_Z = \|P[x-z]\|_Y + t\|P[z]\|_Z \le C_3 \left(\|x-z\|_Y + t\|z\|_Z\right),$$

where $C_3 := \max(C_1, C_2)$. By taking the infinum over $z \in Z$ we therefore have, for all $t > 0$:

$$\inf_{z\in Z} \left(\|x - P[z]\|_Y + t\|P[z]\|_Z\right) \le C_3 \inf_{z\in Z} \left(\|x-z\|_Y + t\|z\|_Z\right) = C_3 K(t,x,Y,Z).$$

Since z only appears as $P[z]$ in the left hand side the infinum can be taken over $z \in Z\cap U$ and hence $K(t,x,U,Z\cap U) \le C_3 K(t,x,Y,Z)$. Therefore

$$0 = C_3 \lim_{t\to 0^+} t^{-\theta} K(t,x,Y,Z) \ge \lim_{t\to 0^+} t^{-\theta} K(t,x,U,Z\cap U) \ge 0,$$

and hence $x \in (U, Z\cap U)_\theta$. Thus, $(Y,Z)_\theta \cap U \subset (U, Z\cap U)_\theta$ and we obtain the result. □

Throughout this dissertation we will be considering functions of varying degrees of regularity; here we introduce the different Banach spaces that will be considered, see also [38]. Let $\beta = (\beta_1, \ldots, \beta_2)$ be a multi-index with $|\beta| = \sum_{i=1}^{n} \beta_i$, then for an open set $U \subset \mathbb{R}^n$ the Hölder spaces are defined for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ as:

$$C^{\alpha}\left(\bar{U}\right) := \left\{ f \in C\left(\bar{U}\right) : \sup_{\substack{x,y\in\bar{U}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} < \infty \right\},$$

$$C^{k,\alpha}\left(\bar{U}\right) := \left\{ f \in C^{k}\left(\bar{U}\right) : D^{\beta} f \in C^{\alpha}\left(\bar{U}\right) \text{ for all } \beta,\ |\beta| = k \right\},$$

where D is the derivative operator on $\mathbb{R}^n$. Here we use that $C^k\left(\bar{U}\right)$ is the space of functions defined on $\bar{U}$ that are k times continuously differentiable in U, with derivatives up to the order k bounded and continuously extendable up to the boundary. The norms on these spaces are:

$$\|f\|_{C^{k,\alpha}(\bar{U})} := \|f\|_{C^{k}(\bar{U})} + \sum_{|\beta|=k} \sup_{\substack{x,y\in\bar{U}\\ x\neq y}} \frac{|D^{\beta}f(x)-D^{\beta}f(y)|}{|x-y|^{\alpha}},$$

where

$$\|f\|_{C^{k}(\bar{U})} := \sum_{|\beta|\leq k} \sup_{x\in\bar{U}} \left| D^{\beta} f(x) \right|.$$

The little-Hölder spaces are closed subspaces of the Hölder spaces; they share the same norm as the Hölder spaces and are defined as:

$$h^{\alpha}\left(\bar{U}\right) := \left\{ f \in C^{\alpha}\left(\bar{U}\right) : \lim_{r\to 0} \sup_{\substack{x,y\in\bar{U}\\ 0<|x-y|<r}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} = 0 \right\},$$

$$h^{k,\alpha}\left(\bar{U}\right) := \left\{ f \in C^{k,\alpha}\left(\bar{U}\right) : D^{\beta} f \in h^{\alpha}\left(\bar{U}\right) \text{ for all } \beta,\ |\beta| = k \right\}.$$

These spaces are able to be extended to a manifold by means of an atlas and, in the case of a manifold with metric, are equipped with the norm:

$$\|u\|_{h^{k,\alpha}(M^n)} = \|u\|_{C^{k}(M^n)} + \sum_{|\beta|=k} \sup_{\substack{\boldsymbol{p},\boldsymbol{q}\in M^n\\ \boldsymbol{p}\neq\boldsymbol{q}}} \frac{|\nabla^{\beta}u(\boldsymbol{p})-\nabla^{\beta}u(\boldsymbol{q})|}{d(\boldsymbol{p},\boldsymbol{q})^{\alpha}}, \tag{3.4}$$

where $d(\cdot,\cdot)$ is the geodesic distance, [22]. Note that when writing the norm we will drop the space the function is over when it is clear. We have the following lemma for the relationship between the norm of a function on the cylinder and its odd and even extensions on the torus.

Lemma 3.1.3. *Fix $\alpha \in (0,1)$ and let $\rho \in h^{0,\alpha}\left(\overline{\mathscr{C}}^n_{R,d}\right)$ then $u_\rho \in h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ and*

$$\|\rho\|_{h^{0,\alpha}} = \|u_\rho\|_{h^{0,\alpha}}.$$

Further if $\rho(\boldsymbol{q},0) = \rho(\boldsymbol{q},d) = 0$ and we set v_ρ to be its odd extension to $\mathscr{T}^n_{R,d}$, then $v_\rho \in h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ and

$$\|\rho\|_{h^{0,\alpha}} \leq \|v_\rho\|_{h^{0,\alpha}} \leq 2\|\rho\|_{h^{0,\alpha}}.$$

Proof. We first note that if $\rho \in C^0\left(\overline{\mathscr{C}}^n_{R,d}\right)$ then $u_\rho \in C^0\left(\mathscr{T}^n_{R,d}\right)$ and $\|u_\rho\|_{C^0} = \|\rho\|_{C^0}$. We now define for $\boldsymbol{p}_a = (\boldsymbol{q}_a, z_a) \in \mathscr{T}^n_{R,d}$, $a \in \{1,2\}$, the point $\bar{\boldsymbol{p}}_a := (\boldsymbol{q}_a, |z_a|) \in \overline{\mathscr{C}}^n_{R,d}$ and seek a bound of the form

$$\frac{|u_\rho(\boldsymbol{p}_1) - u_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \leq \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\bar{\boldsymbol{p}}_2)|}{d(\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2)^\alpha}, \tag{3.5}$$

for all $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathscr{T}^n_{R,d}$. If $z_1, z_2 \in [0,d]$ or $z_1, z_2 \in (-d,0)$ then we have equality, whereas if $z_1 \in [0,d]$ and $z_2 \in (-d,0)$ then $d(\boldsymbol{p}_1,\boldsymbol{p}_2) \geq d(\boldsymbol{p}_1,\bar{\boldsymbol{p}}_2)$ so the bound holds. Therefore:

$$\begin{aligned}
\lim_{r\to 0} \sup_{\substack{\boldsymbol{p}_1,\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|u_\rho(\boldsymbol{p}_1) - u_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} &\leq \lim_{r\to 0} \sup_{\substack{\boldsymbol{p}_1,\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\bar{\boldsymbol{p}}_2)|}{d(\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2)^\alpha} \\
&= \lim_{r\to 0} \sup_{\substack{\boldsymbol{p}_1,\boldsymbol{p}_2 \in \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\boldsymbol{p}_1) - \rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \\
&= 0.
\end{aligned}$$

So $u_\rho \in h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ and the equality of norms follows from taking the supremum in equation (3.5) and from $\|u_\rho\|_{h^{0,\alpha}} \geq \|u_\rho|_{\overline{\mathscr{C}}^n_{R,d}}\|_{h^{0,\alpha}} = \|\rho\|_{h^{0,\alpha}}$.

We now turn to the odd extension and note that if $\rho \in C^0\left(\overline{\mathscr{C}}^n_{R,d}\right)$ and is zero at $z = 0$ then $v_\rho \in C^0\left(\mathscr{T}^n_{R,d}\right)$ and $\|v_\rho\|_{C^0} = \|\rho\|_{C^0}$. In this case we note that if $z_1 \in (0,d)$ and $z_2 \in (-d,0)$ then either the geodesic joining $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ crosses $z = 0$, or $z = d$. Therefore it passes through a point $\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}(\boldsymbol{p}_1,\boldsymbol{p}_2) = (\tilde{\boldsymbol{q}}(\boldsymbol{p}_1,\boldsymbol{p}_2), 0)$, or $\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}(\boldsymbol{p}_1,\boldsymbol{p}_2) = (\tilde{\boldsymbol{q}}(\boldsymbol{p}_1,\boldsymbol{p}_2), d)$. This point can be associated with the corresponding point on the cylinder, hence $\rho(\tilde{\boldsymbol{p}}) = 0$. We therefore obtain:

$$\begin{aligned}
\frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} &= \frac{|\rho(\bar{\boldsymbol{p}}_1) + \rho(\bar{\boldsymbol{p}}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \\
&= \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\tilde{\boldsymbol{p}}) + \rho(\bar{\boldsymbol{p}}_2) - \rho(\tilde{\boldsymbol{p}})|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \\
&\leq \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\tilde{\boldsymbol{p}})|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} + \frac{|\rho(\bar{\boldsymbol{p}}_2) - \rho(\tilde{\boldsymbol{p}})|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \\
&\leq \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\tilde{\boldsymbol{p}})|}{d(\bar{\boldsymbol{p}}_1,\tilde{\boldsymbol{p}})^\alpha} + \frac{|\rho(\bar{\boldsymbol{p}}_2) - \rho(\tilde{\boldsymbol{p}})|}{d(\bar{\boldsymbol{p}}_2,\tilde{\boldsymbol{p}})^\alpha}.
\end{aligned} \tag{3.6}$$

We now fix an $r \in (0, d/2)$ and, due to the symmetry of the domain and v_ρ, we have that:

$$\begin{aligned}
\sup_{\substack{\boldsymbol{p}_1,\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} &= \sup_{\substack{\boldsymbol{p}_1 \in \mathscr{T}^n_{R,d} \\ z_1 \in [0,d]}} \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \\
&= \max \Bigg(\sup_{\boldsymbol{p}_1 \in \overline{\mathscr{C}}^n_{R,d}} \sup_{\substack{\boldsymbol{p}_2 \in \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\boldsymbol{p}_1) - \rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha}, \\
&\qquad \sup_{\substack{\boldsymbol{p}_1 \in \mathscr{T}^n_{R,d} \\ z_1 \in [0,d]}} \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \setminus \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \Bigg).
\end{aligned} \tag{3.7}$$

Now if $z_1 \in \{0, d\}$ then we have that:

$$\begin{aligned}
\sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \setminus \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} &= \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} \setminus \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_2)|}{d(\boldsymbol{p}_1,\bar{\boldsymbol{p}}_2)^\alpha} \\
&\le \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{C}^n_{R,d} \\ 0<d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\boldsymbol{p}_2)|}{d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_2)^\alpha},
\end{aligned} \tag{3.8}$$

where we have used that

$$\{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} : z_2 \in (-d,0), 0 < d(\boldsymbol{p}_1,\boldsymbol{p}_2) < r\} \subset \{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d} : 0 < d(\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2) < r\},$$

since $d(\boldsymbol{p}_1,\boldsymbol{p}_2) = d(\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2)$. Lastly if $z_1 \in (0,d)$ we can use equation (3.6) to conclude:

$$\begin{aligned}
&\sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d},\ z_2 \in (-d,0) \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \\
&\qquad \le \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d},\ z_2 \in (-d,0) \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\tilde{\boldsymbol{p}})|}{d(\bar{\boldsymbol{p}}_1,\tilde{\boldsymbol{p}})^\alpha} + \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d},\ z_2 \in (-d,0) \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_2) - \rho(\tilde{\boldsymbol{p}})|}{d(\bar{\boldsymbol{p}}_2,\tilde{\boldsymbol{p}})^\alpha} \\
&\qquad \le \sup_{\substack{\boldsymbol{p}_3 \in \partial\mathscr{C}^n_{R,d} \\ 0<d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_3)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\boldsymbol{p}_3)|}{d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_3)^\alpha} + \sup_{\substack{\boldsymbol{p}_2 \in \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\boldsymbol{p}_2) - \rho(\tilde{\boldsymbol{p}})|}{d(\boldsymbol{p}_2,\tilde{\boldsymbol{p}})^\alpha} \\
&\qquad \le \sup_{\substack{\boldsymbol{p}_3 \in \partial\mathscr{C}^n_{R,d} \\ 0<d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_3)<r}} \frac{|\rho(\bar{\boldsymbol{p}}_1) - \rho(\boldsymbol{p}_3)|}{d(\bar{\boldsymbol{p}}_1,\boldsymbol{p}_3)^\alpha} + \sup_{\substack{\boldsymbol{p}_2 \in \overline{\mathscr{C}}^n_{R,d}, \boldsymbol{p}_3 \in \partial\mathscr{C}^n_{R,d} \\ 0<d(\boldsymbol{p}_3,\boldsymbol{p}_2)<r}} \frac{|\rho(\boldsymbol{p}_2) - \rho(\boldsymbol{p}_3)|}{d(\boldsymbol{p}_2,\boldsymbol{p}_3)^\alpha}.
\end{aligned}$$

Therefore:

$$\sup_{\substack{\boldsymbol{p}_1 \in \mathscr{T}^n_{R,d} \\ z_1 \in (0,d)}} \sup_{\substack{\boldsymbol{p}_2 \in \mathscr{T}^n_{R,d},\ z_2 \in (-d,0) \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|v_\rho(\boldsymbol{p}_1) - v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha} \le 2 \sup_{\boldsymbol{p}_1 \in \overline{\mathscr{C}}^n_{R,d}} \sup_{\substack{\boldsymbol{p}_2 \in \overline{\mathscr{C}}^n_{R,d} \\ 0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}} \frac{|\rho(\boldsymbol{p}_1) - \rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha},$$

and hence by combining with equations (3.7) and (3.8) we have that:

$$\sup_{\substack{\boldsymbol{p}_1,\boldsymbol{p}_2\in\mathscr{T}^n_{R,d}\\0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}}\frac{|v_\rho(\boldsymbol{p}_1)-v_\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha}\leq 2\sup_{\substack{\boldsymbol{p}_1,\boldsymbol{p}_2\in\overline{\mathscr{C}}^n_{R,d}\\0<d(\boldsymbol{p}_1,\boldsymbol{p}_2)<r}}\frac{|\rho(\boldsymbol{p}_1)-\rho(\boldsymbol{p}_2)|}{d(\boldsymbol{p}_1,\boldsymbol{p}_2)^\alpha}.$$

Taking the limit $r\to 0$ gives that $v_\rho\in h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$. The bound on the norm also follows from equation (3.6). ☐

Corollary 3.1.4. *Fix $l\in\{1,2\}$, $\alpha\in(0,1)$ and let $\rho\in h^{l,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$ then its even extension u_ρ is in $h^{l,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ and*

$$\|\rho\|_{h^{l,\alpha}}\leq\|u_\rho\|_{h^{l,\alpha}}\leq 2\|\rho\|_{h^{l,\alpha}}.$$

Proof. We first note that if $\rho\in C^l_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, $l\in\{1,2\}$, then $u_\rho\in C^l\left(\mathscr{T}^n_{R,d}\right)$ and $\|u_\rho\|_{C^l}=\|\rho\|_{C^l}$. When defined, we have the derivatives given by:

$$\nabla_i u_\rho(\boldsymbol{q},z)=\begin{cases}\nabla_i\rho(\boldsymbol{q},z) & z\in[0,d],\\ \nabla_i\rho(\boldsymbol{q},-z) & z\in(-d,0),\end{cases}\quad i\neq n,$$

$$\nabla_n u_\rho=\begin{cases}\nabla_n\rho(\boldsymbol{q},z) & z\in[0,d],\\ -\nabla_n\rho(\boldsymbol{q},-z) & z\in(-d,0),\end{cases}$$

and

$$\nabla_i\nabla_j u_\rho(\boldsymbol{q},z)=\begin{cases}\nabla_i\nabla_j\rho(\boldsymbol{q},z) & z\in[0,d],\\ \nabla_i\nabla_j\rho(\boldsymbol{q},-z) & z\in(-d,0),\end{cases}\quad i,j\neq n\text{ or }i=j=n,$$

$$\nabla_i\nabla_j u_\rho(\boldsymbol{q},z)=\begin{cases}\nabla_i\nabla_j\rho(\boldsymbol{q},z) & z\in[0,d],\\ -\nabla_i\nabla_j\rho(\boldsymbol{q},-z) & z\in(-d,0),\end{cases}\quad i\text{ or }j=n.$$

Since all these functions are either even or odd, by Lemma 3.1.3 we get the result. ☐

The interpolation functors allow characterisation of the little-Hölder spaces in terms of the continuous function spaces:

$$h^{l\theta}\left(\bar{U}\right)=\left(C\left(\bar{U}\right),C^l\left(\bar{U}\right)\right)_\theta,\tag{3.9}$$

for $l\in\mathbb{N}$ and $\theta\in(0,1)$ such that $l\theta\notin\mathbb{N}$, [38]. Here we use the notation that $h^\sigma\left(\bar{U}\right)=h^{\lfloor\sigma\rfloor,(\sigma-\lfloor\sigma\rfloor)}\left(\bar{U}\right)$, for a real number $\sigma\in\mathbb{R}$. The Reiteration Theorem 3.1.1 then gives the following corollary.

Corollary 3.1.5. *For any $l\in\mathbb{N}_0$ such that $\theta_0\left(l+\theta_2-\theta_1\right)+\theta_1\notin\mathbb{N}$:*

$$\left(h^{0,\theta_1}\left(\bar{U}\right),h^{l,\theta_2}\left(\bar{U}\right)\right)_{\theta_0}=h^{\theta_0(l+\theta_2-\theta_1)+\theta_1}\left(\bar{U}\right).\tag{3.10}$$

The above theorem can be extended to little-Hölder spaces on manifolds without boundary, see for example equation 19 in [26].

3.2 Sectorial Operators

A linear operator, $A : Z \subset Y \to Y$, is called *sectorial* if there exist $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $\omega \in \mathbb{R}$ and $M > 0$ such that

(i) $\rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,

(ii) $\|R(\lambda, A)\|_{\mathcal{L}(Y,Y)} \leq \frac{M}{|\lambda-\omega|}$ for all $\lambda \in S_{\theta,\omega}$,

here $\rho(A)$ is the resolvent set, $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator and $\|\cdot\|_{\mathcal{L}(Y,Y)}$ is the standard linear operator norm, [38].

We also have the following lemma from [38] that gives a sufficient condition for an operator to be sectorial.

Proposition 3.2.1 (Proposition 2.1.11 [38])**.** *Let $A : Z \subset Y \to Y$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C} : Re(\lambda) \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(Y,Y)} \leq M, \quad Re(\lambda) \geq \omega, \tag{3.11}$$

with $\omega \in \mathbb{R}$, $M > 0$. Then A is sectorial.

We also have a different characterisation:

Lemma 3.2.2. *Assume that $\rho(A)$ contains the half plane $\{\lambda \in \mathbb{C} : Re(\lambda) \geq \omega\}$, then the condition (3.11) is equivalent to*

$$\|z\|_Z \leq \kappa \|(\lambda I - A)[z]\|_Y, \quad \textit{for all } z \in Z, \quad Re(\lambda) \geq \omega, \tag{3.12}$$

for some $\kappa > 0$.

Proof. If (3.12) holds, then we obtain the bound

$$\begin{aligned}
\|\lambda z\|_Y &= \|(\lambda I - A)[z] + A[z]\|_Y \\
&\leq \|(\lambda I - A)[z]\|_Y + \|A\|_{\mathcal{L}(Z,Y)} \|z\|_Z \\
&\leq (1 + \kappa \|A\|_{\mathcal{L}(Z,Y)}) \|(\lambda I - A)[z]\|_Y,
\end{aligned}$$

for all $z \in Z$, which gives us (3.11).

Alternatively we wish to bound $\|z\|_Z$, assuming (3.11). For $Re(\lambda) \geq \omega$ we have:

$$\begin{aligned}
\|z\|_Z &= \|R(\omega, A) \left[(\lambda I - A)[z] + (\omega - \lambda) z\right] \|_Z \\
&\leq \|R(\omega, A)\|_{\mathcal{L}(Y,Z)} \|(\lambda I - A)[z] + (\omega - \lambda) z\|_Y \\
&\leq \|R(\omega, A)\|_{\mathcal{L}(Y,Z)} \left(\|(\lambda I - A)[z]\|_Y + \left|\frac{\omega}{\lambda} - 1\right| \|\lambda z\|_Y \right) \\
&\leq \|R(\omega, A)\|_{\mathcal{L}(Y,Z)} \left(1 + 2M\right) \|(\lambda I - A)[z]\|_Y
\end{aligned}$$

□

We now assume Z and Y are Banach spaces with Z dense in Y. As defined by Amann in [3] we let:

$$\mathcal{H}(Z,Y) := \{A \in \mathcal{L}(Z,Y) : \mathcal{G}(A) \text{ is a strongly continuous analytic semigroup}\},$$

where $\mathcal{G}(A) = \{e^{-tA} : t \geq 0\}$. This space can be seen to be equivalent to the space of sectorial operators. Firstly, by Proposition 2.1.4 in [38], see also Remark 2.1.5, if $A : Z \to Y$ is sectorial then $\mathcal{G}(-A)$ is a strongly continuous analytic semigroup and hence $-A \in \mathcal{H}(Z,Y)$ (in fact $\mathcal{G}(-A)$ is strongly continuous if and only if Z is dense in Y). The reverse implication follows by combining Proposition 3.2.1 with the following theorem:

Theorem 3.2.3 (Theorem 1.2.2 [3]). *$A \in \mathcal{H}(Z,Y)$ if and only if there exist $\kappa \geq 1$ and $\omega > 0$ such that $\omega I + A$ is an isometry from Z to Y and*

$$\kappa^{-1} \leq \frac{\|(\lambda I + A)[z]\|_Y}{|\lambda|\|z\|_Y + \|z\|_Z} \leq \kappa, \quad \textit{for all } z \in Z, \quad Re(\lambda) \geq \omega.$$

We now introduce the Shauder estimates on the Hölder spaces. These will be used in Theorem 3.2.6 to determine a class of sectorial operators.

Theorem 3.2.4 (Theorem 27 (a) [15]). *Let A be a linear, elliptic differential operator of order k on a manifold, M^n, that is compact without boundary. Given a constant $\alpha \in (0,1)$ and an integer $l \geq 0$ there are constants c_1, c_2, c_3 such that for every $v \in C^{k+l,\alpha}(M^n)$,*

$$\|v\|_{C^{k+l,\alpha}} \leq c_1\|A[v]\|_{C^{l,\alpha}} + c_2\|v\|_{C^0} \leq c_3\|v\|_{C^{k+l,\alpha}}.$$

Moreover, if one restricts v so that it is orthogonal (in $L^2(M^n)$) to the nullspace of A, then we can let $c_2 = 0$ (with a new constant c_1).

Another standard theorem of elliptic operators that we require is the following:

Theorem 3.2.5 (Theorem 37 [15]). *Let A be a linear, uniformly elliptic differential operator of order k on a manifold, M^n, that is compact without boundary. The eigenvalues of A are discrete, having a limit point only at infinity.*

These estimates allow us to prove that elliptic operators are sectorial as maps into the Hölder or little-Hölder spaces.

Theorem 3.2.6. *Let $k, l \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $A : h^{k+l,\alpha}(M^n) \to h^{l,\alpha}(M^n)$ (or from $C^{k+l,\alpha}(M^n)$ to $C^{l,\alpha}(M^n)$) be a linear, uniformly elliptic differential operator of order k, where M^n is compact without boundary. Then $-A$ is sectorial.*

Proof. First we note that due to the compact embedding $h^{k+l,\alpha}(M^n) \subset h^{l,\alpha}(M^n)$ the spectrum of A consists entirely of eigenvalues. By Theorem 3.2.5 there exists ω such that, if λ is any eigenvalue of A, then $\mathrm{Re}(\lambda) > -\omega$ and hence $\lambda I + A$ is a linear isomorphism for all $\mathrm{Re}(\lambda) \geq \omega$. Therefore by Theorem 3.2.4 (since the little-Hölder norms are the same as the Hölder norms) we obtain the bound:

$$\|v\|_{h^{k+l,\alpha}} \leq c_1 \| (\lambda I + A)[v] \|_{h^{l,\alpha}}, \tag{3.13}$$

for all $v \in h^{k+l,\alpha}(M^n)$ and $\mathrm{Re}(\lambda) \geq \omega$. Hence by Lemma 3.2.2 and Proposition 3.2.1 we have that $-A$ is sectorial. The proof for the Hölder spaces is the same. □

Another important property of sectorial operators is the fact that they remain sectorial under certain perturbations, see the following two propositions.

Proposition 3.2.7 (Proposition 2.4.1 [38])**.** *Let $\theta \in (0,1)$ and $A : Z \to Y$ be sectorial. Then:*

- *If $B \in \mathcal{L}((Y,Z)_\theta, Y)$ then $A+B : Z \to Y$ is sectorial. This remains true if $\theta = 0$.*
- *If $B \in \mathcal{L}(Z, (Y,Z)_\theta)$ then $A+B : Z \to Y$ is sectorial. This remains true if $\theta = 1$.*

Proposition 3.2.8 (Proposition 2.4.2 [38])**.** *Let $A : Z \to Y$ be sectorial with constants ω, θ, M, and let $B \in \mathcal{L}(Z,Y)$, with $\|B\|_{\mathcal{L}(Z,Y)} < \frac{1}{M+1}$. Then $A+B : Z \to Y$ is sectorial.*

4

Existence in Interpolation Spaces

In this chapter we analyse the existence of solutions to equation (1.5) for initial conditions in the interpolation spaces. The calculations are carried out in the little-Hölder spaces so that solutions are continuous in time at $t = 0$; however similar results are valid for functions in the Hölder spaces. We begin the calculations by linearising the speed of the height function at the base hypersurface. This is then shown to be a sectorial operator on the interpolation spaces and existence in these spaces can be proven. For the case of the volume preserving flow the flow is quasilinear, which allows for improvement in the regularity for times greater than zero.

4.1 Linearisation

To analyse the flow in equation (1.5) we will consider its linearisation about the base hypersurface. The speed operator is given by:

$$G(\rho) := L(\rho)\left(h(\rho) - F(\boldsymbol{\kappa}_\rho)\right), \tag{4.1}$$

where $h(\rho) := \frac{1}{\int_{M^n} \Xi(\boldsymbol{\kappa}_\rho)\, d\mu_\rho} \int_{M^n} F(\boldsymbol{\kappa}_\rho)\, \Xi(\boldsymbol{\kappa}_\rho)\, d\mu_\rho$ and $L(\rho) := \sqrt{1 + \left|\tilde{\nabla}\rho\right|^2}$. We first turn our attention to the global part of the equation.

Lemma 4.1.1. *For a constant principal curvatures hypersurface $\Omega_0 \subset \mathbb{R}^{n+1}$, i.e. where $\boldsymbol{\kappa}_0(\boldsymbol{p}) = \boldsymbol{\kappa}_0$ for all $\boldsymbol{p} \in M^n$, the linearisation of the weighted average curvature function about $\rho = 0 \in C^2(M^n)$ is:*

$$\partial h(0)[v] = \fint_{M^n} \partial F(\boldsymbol{\kappa}_\rho)\big|_{\rho=0}[v]\, d\mu_0,$$

for $v \in C^2(M^n)$.

Proof. For a Fréchet differentiable f we have:

$$\partial\left(\int_{M^n} f\, d\mu_0\right)[v] = \int_{M^n} \partial f[v]\, d\mu_0,$$

so calculating we obtain:

$$\begin{aligned}
\partial h\,(0)\,[v] = &\,\partial\left(\frac{1}{\int_{M^n} \Xi(\boldsymbol{\kappa}_\rho)\mu\,(\rho)\; d\mu_0} \int_{M^n} \Xi\,(\boldsymbol{\kappa}_\rho)\, F\,(\boldsymbol{\kappa}_\rho)\, \mu\,(\rho)\; d\mu_0\right)\Bigg|_{\rho=0} [v] \\
=&\frac{1}{\int_{M^n} \Xi\,(\boldsymbol{\kappa}_0)\; d\mu_0}\, \partial\left(\int_{M^n} \Xi\,(\boldsymbol{\kappa}_\rho)\, F\,(\boldsymbol{\kappa}_\rho)\, \mu\,(\rho)\; d\mu_0\right)\Bigg|_{\rho=0} [v] \\
&- \frac{\int_{M^n} \Xi\,(\boldsymbol{\kappa}_0)\, F\,(\boldsymbol{\kappa}_0)\; d\mu_0}{\left(\int_{M^n} \Xi\,(\boldsymbol{\kappa}_0)\; d\mu_0\right)^2}\, \partial\left(\int_{M^n} \Xi\,(\boldsymbol{\kappa}_\rho)\, \mu\,(\rho)\; d\mu_0\right)\Bigg|_{\rho=0} [v] \\
=&\frac{1}{\int_{M^n} \Xi\,(\boldsymbol{\kappa}_0)\; d\mu_0}\left(\int_{M^n} \partial\left(\Xi\,(\boldsymbol{\kappa}_\rho)\, \mu\,(\rho)\, F\,(\boldsymbol{\kappa}_\rho)\right)\big|_{\rho=0} [v]\, d\mu_0\right. \\
&\qquad\qquad \left. -F\,(\boldsymbol{\kappa}_0) \int_{M^n} \partial\left(\Xi\,(\boldsymbol{\kappa}_\rho)\, \mu\,(\rho)\right)\big|_{\rho=0} [v]\, d\mu_0\right) \\
=&\frac{1}{\int_{M^n} \Xi\,(\boldsymbol{\kappa}_0)\; d\mu_0} \int_{M^n} \Xi\,(\boldsymbol{\kappa}_0)\; \partial F\,(\boldsymbol{\kappa}_\rho)\big|_{\rho=0} [v]\, d\mu_0.
\end{aligned}$$

The lemma follows since $\Xi\,(\boldsymbol{\kappa}_0)$ is constant over M^n. $\square$

Importantly, we see that the linearisation of the speed does not depend on the weight function used when averaging the curvature function. This allows us to treat all the weighted volume preserving curvature flows at once. Using the chain rule the linearisation of the curvature function can be written as $\partial F(\boldsymbol{\kappa}_\rho)\big|_{\rho=0} = \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\boldsymbol{\kappa}_0)\partial\kappa_a\,(0)$. To proceed we need the following lemma:

Lemma 4.1.2. *Let $\kappa\,(\rho)$ be a principal curvature of the hypersurface Ω_ρ with corresponding unit (with respect to the Ω_0 metric) principal direction $(\zeta_\rho)^i$, then:*

$$\partial\kappa\,(0) = -\mathring{\zeta}^i \mathring{\zeta}^j \nabla_i \nabla_j - \kappa(0)^2,$$

where $\mathring{\zeta}^i := (\zeta_0)^i$.

Proof. We start by noting that the condition $\mathring{g}_{ij}(\zeta_\rho)^i(\zeta_\rho)^j = 1$ implies:

$$\mathring{g}_{ij}\mathring{\zeta}^j\; \partial(\zeta_\rho)^i\big|_{\rho=0} = 0. \tag{4.2}$$

Next, from the definition of $\kappa\,(\rho)$ we have that $(g_\rho)^{il}(h_\rho)_{lj}(\zeta_\rho)^j = \kappa\,(\rho)\,(\zeta_\rho)^i$ so, by linearising about $\rho = 0$, we obtain:

$$\mathring{\zeta}^j \mathring{h}_{lj}\; \partial(g_\rho)^{il}\Big|_{\rho=0} + \mathring{\zeta}^j \mathring{g}^{il}\; \partial(h_\rho)_{lj}\big|_{\rho=0} + \mathring{g}^{il}\mathring{h}_{lj}\; \partial(\zeta_\rho)^j\big|_{\rho=0} = \mathring{\zeta}^i \partial\kappa\,(0) + \kappa(0)\; \partial(\zeta_\rho)^i\big|_{\rho=0}\,. \tag{4.3}$$

Multiplying this equation by $\mathring{g}_{ik}\mathring{\zeta}^k$ as well as using that $\mathring{\zeta}^l\mathring{h}_{lj} = \kappa(0)\mathring{g}_{lj}\mathring{\zeta}^l$ and (4.2), we obtain:

$$\begin{aligned}
\partial\kappa\,(0) &= \mathring{g}_{ik}\mathring{\zeta}^k\mathring{\zeta}^j\mathring{h}_{lj}\;\partial(g_\rho)^{il}\Big|_{\rho=0} + \mathring{g}_{ik}\mathring{\zeta}^k\mathring{\zeta}^j\mathring{g}^{il}\;\partial(h_\rho)_{lj}\big|_{\rho=0} + \mathring{g}_{ik}\mathring{\zeta}^k\mathring{g}^{il}\mathring{h}_{lj}\;\partial(\zeta_\rho)^j\big|_{\rho=0} \\
&= -\mathring{g}_{ik}\mathring{\zeta}^k\mathring{\zeta}^j\mathring{h}_{lj}\mathring{g}^{ip}\mathring{g}^{ql}\;\partial(g_\rho)_{pq}\big|_{\rho=0} + \mathring{\zeta}^l\mathring{\zeta}^j\;\partial(h_\rho)_{lj}\big|_{\rho=0} + \mathring{\zeta}^l\mathring{h}_{lj}\;\partial(\zeta_\rho)^j\big|_{\rho=0} \\
&= -\mathring{\zeta}^p\mathring{\zeta}^j\mathring{h}_j^q\;\partial(g_\rho)_{pq}\big|_{\rho=0} + \mathring{\zeta}^l\mathring{\zeta}^j\;\partial(h_\rho)_{lj}\big|_{\rho=0} + \kappa(0)\mathring{g}_{lj}\mathring{\zeta}^l\;\partial(\zeta_\rho)^j\big|_{\rho=0} \\
&= \mathring{\zeta}^i\mathring{\zeta}^j\left(\partial(h_\rho)_{ij}\big|_{\rho=0} - \mathring{h}_j^q\;\partial(g_\rho)_{iq}\big|_{\rho=0}\right).
\end{aligned}$$

We use the second fundamental form for a normal graph given in (2.12):

$$\begin{aligned}
(h_\rho)_{ij} =& L(\rho)^{-1}\left(\mathring{h}_i^l\left(\mathring{g}_{lj} + \rho\mathring{h}_{lj}\right) - \nabla_i\nabla_j\rho\right) \\
&+ L(\rho)^{-1}(\tilde{g}_\rho)^{kp}\left(\delta_p^l + \rho\mathring{h}_p^l\right)\left(\mathring{h}_{jl}\nabla_i\rho + \mathring{h}_{il}\nabla_j\rho + \rho\nabla_l\mathring{h}_{ij}\right)\nabla_k\rho.
\end{aligned}$$

In order to calculate the linearisation at $\rho = 0$ we note that $\partial L(0) = 0$ and $L(0) = 1$, hence the $L(\rho)^{-1}$ factor does not affect the linearisation. Also note that the last term is second order in ρ, so it also vanishes when taking the linearisation at $\rho = 0$. The linearisation at $\rho = 0$ is then easily found to be:

$$\partial(h_\rho)_{ij}\big|_{\rho=0} = -\nabla_i\nabla_j + \mathring{h}_i^l\mathring{h}_{lj}. \tag{4.4}$$

From the formula for the metric given in (2.9) we have that $\partial(g_\rho)_{iq}\big|_{\rho=0} = 2\mathring{h}_{iq}$, so:

$$\partial\kappa(0) = \mathring{\zeta}^i\mathring{\zeta}^j\left(-\nabla_i\nabla_j - \mathring{h}_{il}\mathring{h}_j^l\right) \tag{4.5}$$

The result then follows from $\mathring{\zeta}^j\mathring{h}_{il}\mathring{h}_j^l = \kappa(0)\mathring{h}_{il}\mathring{\zeta}^l = \kappa(0)^2\mathring{g}_{il}\mathring{\zeta}^l$ and because $\mathring{\zeta}$ is a unit vector. □

Combining these results, we are able to give the full linearisation of the speed operator at a hypersurface of constant principal curvatures.

Proposition 4.1.3. *Let Ω_0 be a hypersurface with constant principal curvatures and $\mathring{\zeta}_a$ be the unit principal direction vector corresponding to the principal curvature $\kappa_a(0)$, i.e. $\mathring{h}_i^j\mathring{\zeta}_a^i = \kappa_a(0)\mathring{\zeta}_a^j$ (where we do not sum over a). Then:*

$$\begin{aligned}
\partial G\,(0)\,[v] =& \sum_{a=1}^n \frac{\partial F}{\partial\kappa_a}\,(\boldsymbol{\kappa}_0)\left(\mathring{\zeta}_a^i\mathring{\zeta}_a^j\nabla_i\nabla_j + \kappa_a(0)^2\right)v \\
&- \sum_{a=1}^n \frac{\partial F}{\partial\kappa_a}\,(\boldsymbol{\kappa}_0)\fint_{M^n}\left(\mathring{\zeta}_a^i\mathring{\zeta}_a^j\nabla_i\nabla_j + \kappa_a(0)^2\right)v\,d\mu_0,
\end{aligned}$$

for $v \in C^2\,(M^n)$.

We can simplify this expression in the case that Ω_0 is a sphere using the fact that all principal curvatures are equal, so $\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0) = \frac{\partial F}{\partial \kappa_a}(\boldsymbol{\kappa}_0)$ for all $a = 1, \ldots, n$. We also use the divergence theorem to remove the derivatives from the global term.

Corollary 4.1.4. *The linearisation of (4.1) at $\rho = 0 \in C^2(\mathscr{S}^n_R)$ is given by:*

$$\partial G_s(0)[v] = \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\left(\left(\Delta_{\mathscr{S}^n_R} + \frac{n}{R^2}\right)v - \frac{n}{R^2}\fint_{\mathscr{S}^n_R} v\, d\mu_0\right), \tag{4.6}$$

for $v \in C^2(\mathscr{S}^n_R)$.

For $u \in C^2\left(\mathscr{T}^n_{R,d}\right)$ we set

$$G_t(u) := \sqrt{1 + \left|\tilde{\nabla} u\right|^2}\left(\frac{1}{\int_{\mathscr{T}^n_{R,d}} \Xi(\boldsymbol{\kappa}_u)\, d\mu_u}\int_{\mathscr{T}^n_{R,d}} F(\boldsymbol{\kappa}_u)\,\Xi(\boldsymbol{\kappa}_u)\, d\mu_u - F(\boldsymbol{\kappa}_u)\right) \tag{4.7}$$

and the result of Proposition 4.1.3 is still applicable, with $\kappa_a(0)$ and $\mathring{\zeta}_a$ given by even extensions of the principal curvatures and directions on the cylinder. We order $\boldsymbol{\kappa}_u$ such that $\kappa_n(0) = 0$, and hence $\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0) = \frac{\partial F}{\partial \kappa_i}(\boldsymbol{\kappa}_0)$ for all $i = 1, \ldots, n-1$.

Corollary 4.1.5. *The linearisation of (4.7) at $u = 0 \in C^2\left(\mathscr{T}^n_{R,d}\right)$ is:*

$$\partial G_t(0)[v] = \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\left(\Delta_{\mathscr{S}^{n-1}_R} + \frac{n-1}{R^2}\right)v + \frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)\frac{\partial^2 v}{\partial z^2} - \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n-1}{R^2}\fint_{\mathscr{T}^n_{R,d}} v\, d\mu_0, \tag{4.8}$$

for $v \in C^2\left(\mathscr{T}^n_{R,d}\right)$.

4.2 A Sectorial Operator

In this section we will prove an important property of the linearisations $\partial G_s(\rho)$ and $\partial G_t(u)$ for ρ, u in a neighbourhood of zero. We show that for each of these operators there exists a sectorial operator $A : h^{2,\alpha_0}(M^n) \to h^{0,\alpha_0}(M^n)$ such that the original operator is the part of A in $h^{0,\alpha}(M^n)$, $\alpha \in (\alpha_0, 1)$, which is an interpolation space by equation (3.10). More precisely we have the following lemmas:

Lemma 4.2.1. *For any $0 < \alpha < 1$ and $0 < \alpha_0 < \alpha$ there exists a neighbourhood, $O_{s,1}$, of $0 \in h^{2,\alpha}(\mathscr{S}^n_R)$ such that the operator $\partial G_s(\rho) : h^{2,\alpha}(\mathscr{S}^n_R) \to h^{0,\alpha}(\mathscr{S}^n_R)$ is the part in $h^{0,\alpha}(\mathscr{S}^n_R)$ of a sectorial operator $A_\rho : h^{2,\alpha_0}(\mathscr{S}^n_R) \to h^{0,\alpha_0}(\mathscr{S}^n_R)$ for all $\rho \in O_{s,1}$.*

Proof. We start by fixing α and choosing any α_0 such that $0 < \alpha_0 < \alpha$. We next define the functional $\bar{G}_s : h^{2,\alpha_0}\left(\mathscr{S}^n_R\right) \to h^{0,\alpha_0}\left(\mathscr{S}^n_R\right)$:

$$\bar{G}_s\left(\rho\right) := L(\rho)\left(h\left(\rho\right) - F\left(\boldsymbol{\kappa}_\rho\right)\right), \tag{4.9}$$

so that if we set $A_\rho = \partial\bar{G}_s(\rho)$ it is clear that $\partial G(\rho)$ is the part in $h^{0,\alpha}\left(\mathscr{S}^n_R\right)$ of A_ρ. It remains to prove that A_ρ is sectorial for $\rho \in O$. To do this we use equation (4.6) to calculate A_0:

$$A_0[v] = \frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right)\left(\left(\Delta_{\mathscr{S}^n_R} + \frac{n}{R^2}\right)v - \frac{n}{R^2}\fint_{\mathscr{S}^n_R} v\, d\mu_0\right). \tag{4.10}$$

Since we have that $\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)$ is positive, the operator $-\tilde{A}_s$, where

$$\tilde{A}_s := \frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right)\left(\Delta_{\mathscr{S}^n_R} + \frac{n}{R^2}\right), \tag{4.11}$$

is uniformly elliptic and hence $\tilde{A}_s : h^{2,\alpha_0}\left(\mathscr{S}^n_R\right) \to h^{\alpha_0}\left(\mathscr{S}^n_R\right)$ is sectorial, by Theorem 3.2.6. Also the map

$$v \to -\frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right)\frac{n}{R^2}\fint_{\mathscr{S}^n_R} v\, d\mu_0 \tag{4.12}$$

is in $\mathcal{L}\left(h^{2,\alpha_0}\left(\mathscr{S}^n_R\right), h^{2,\alpha_0}\left(\mathscr{S}^n_R\right)\right)$ so by Proposition 3.2.7 we have that A_0 is sectorial. This then implies by Proposition 3.2.8 that $A_\rho = A_0 + \left(\partial\bar{G}_s(\rho) - \partial\bar{G}_s(0)\right)$ is sectorial for all ρ in a neighbourhood of zero, $O_{s,2} \subset h^{2,\alpha_0}\left(\mathscr{S}^n_R\right)$. By setting $O_{s,1} = O_{s,2} \cap h^{2,\alpha}\left(\mathscr{S}^n_R\right)$ we finish the proof. $\square$

Lemma 4.2.2. *For any $0 < \alpha < 1$ and $0 < \alpha_0 < \alpha$ there exists a neighbourhood, $O_{t,1}$, of $0 \in h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ such that the operator $\partial G_t(u) : h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right) \to h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ is the part in $h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ of a sectorial operator $A_u : h^{2,\alpha_0}\left(\mathscr{T}^n_{R,d}\right) \to h^{0,\alpha_0}\left(\mathscr{T}^n_{R,d}\right)$ for all $u \in O_{t,1}$.*

Proof. The proof follows the same reasoning as in Lemma 4.2.1. We give here only the differences in the proof. Firstly

$$\tilde{A}_t = \frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right)\left(\Delta_{\mathscr{S}^{n-1}_R} + \frac{n-1}{R^2}\right) + \frac{\partial F}{\partial \kappa_n}\left(\boldsymbol{\kappa}_0\right)\frac{\partial^2}{\partial z^2}. \tag{4.13}$$

Here again, $-\tilde{A}_t$ is uniformly elliptic, since $\frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right), \frac{\partial F}{\partial \kappa_n}\left(\boldsymbol{\kappa}_0\right) > 0$. Secondly the factor in front of the global term is different, however this does not affect the calculations. $\square$

4.3 Existence

We are now able to obtain short time existence for the weighted volume preserving curvature flow in equation (1.4) with an initial hypersurface that is a graph over a sphere or cylinder with small height function. We will be using Theorem 8.4.1 in [38], which we restate with some simplifications:

Theorem 4.3.1. *Let $G : O \subset h^{2,\alpha}(M^n) \to h^{0,\alpha}(M^n)$, $\alpha \in (0,1)$, be such that G and ∂G are continuous in O and for every $\bar{v} \in O$ the operator $\partial G(\bar{v})$ is the part in $h^{0,\alpha}(M^n)$ of a sectorial operator $A : h^{2,\alpha_0}(M^n) \to h^{0,\alpha_0}(M^n)$, $\alpha_0 \in (0,\alpha)$. Then for every $\bar{v} \in O$ there are $\delta, r > 0$, such that if $\|v_0 - \bar{v}\|_{h^{2,\alpha}} \leq r$, then the problem:*

$$v'(t) = G(v(t)),\ 0 \leq t < \delta,\ v(0) = v_0,$$

has a unique maximal solution $v \in C\left([0,\delta), h^{2,\alpha}(M^n)\right) \cap C^1\left([0,\delta), h^{0,\alpha}(M^n)\right)$.

We now prove existence for hypersurfaces close to a sphere. This result, for the case of mixed volume preserving flows, has been included in the paper [27].

Theorem 4.3.2. *There exist $\delta, r > 0$ such that for any function $\rho_0 \in h^{2,\alpha}(\mathscr{S}^n_R)$ satisfying $\|\rho_0\|_{h^{2,\alpha}} \leq r$ the equation (1.5), with $M^n = \mathscr{S}^n_R$, has a unique maximal solution:*

$$\rho \in C\left([0,\delta), h^{2,\alpha}(\mathscr{S}^n_R)\right) \cap C^1\left([0,\delta), h^{0,\alpha}(\mathscr{S}^n_R)\right).$$

Moreover, the graph over a sphere Ω_{ρ_0} has a weighted volume preserving curvature flow for $t \in [0,\delta)$, which is given, up to a tangential diffeomorphism, by $\Omega_{\rho(t)}$.

Proof. As in the remark following Condition 4.2 in [6]: since F and Ξ are smooth, symmetric functions of the principal curvatures they are also smooth functions of the elementary symmetric functions, which depend smoothly on the components of the Weingarten map. It is easily seen that the Weingarten map depends smoothly on $\rho \in U_{2,\alpha}$, note that $U_{2,\alpha}$ is defined in (1.12). Therefore G_s depends smoothly on $\rho \in O_{s,3} \subset U_{2,\alpha}$, where the choice of $O_{s,3}$ is such that if $\rho \in O_{s,3}$ then $\int_{\mathscr{S}^n_R} \Xi(\boldsymbol{\kappa}_\rho)\, d\mu_\rho > 0$. The sectorial condition was established in Lemma 4.2.1 for a neighbourhood $O_{s,1}$, so the proof is complete by using Theorem 4.3.1 with $O = O_{s,1} \cap O_{s,3}$ and $\bar{v} = 0$. $\square$

In order to obtain existence for the flow of graphs over cylinders, we first use the same arguments to obtain an existence theorem for the PDE (1.7):

Theorem 4.3.3. *There exists $\delta, r > 0$ such that if u_0 satisfies $\|u_0\|_{h^{2,\alpha}} \leq r$ then (1.7) has a unique maximal solution $u \in C\left([0,\delta), h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)\right) \cap C^1\left([0,\delta), h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)\right)$.*

Since $\|u_{\rho_0}\|_{h^{2,\alpha}}$ is controlled by $\|\rho_0\|_{h^{2,\alpha}}$, see Corollary 3.1.4, and a solution to (1.7) with initial condition u_{ρ_0}, restricted to $\overline{\mathscr{C}}^n_{R,d}$, is a solution of (1.5) we obtain the following corollary:

Corollary 4.3.4. *There exists $\delta, r > 0$ such that for any function $\rho_0 \in h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$ satisfying $\|\rho_0\|_{h^{2,\alpha}} \leq r$ the equation (1.5), with $M^n = \mathscr{C}^n_{R,d}$, has a unique maximal solution:*

$$\rho \in C\left([0,\delta), h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)\right) \cap C^1\left([0,\delta), h^{0,\alpha}\left(\overline{\mathscr{C}}^n_{R,d}\right)\right).$$

Moreover, the graph over a cylinder Ω_{ρ_0} has a weighted volume preserving curvature flow for $t \in [0,\delta)$, which is given, up to a tangential diffeomorphism, by $\Omega_{\rho(t)}$.

4.4 Improvements for Volume Preserving Mean Curvature Flow

In this section we consider the volume preserving mean curvature flow for graphs over cylinders. While the results in Section 4.3 are still valid, we can improve upon them by using the fact that the flow is quasilinear. In place of Theorem 4.3.1, we are able to apply Theorem 12.1 in [2] (see also Theorem 2.11 in [7]), which has a less strict regularity condition for the initial function. This work has been included in [28].

Theorem 4.4.1 (Theorem 12.1 [2])**.** *Suppose that $0 < \gamma < \alpha < \beta < 1$, that $O_{k+1,\alpha}$ is open in $h^{k+1,\alpha}(M^n)$, and that*

$$(Q,f) \in C^{0,1}\left(O_{k+1,\alpha}, \mathcal{H}\left(h^{k+2,\gamma}(M^n), h^{k,\gamma}(M^n)\right) \times h^{k,\alpha}(M^n)\right).$$

Then, for each $v_0 \in O_{k+1,\beta} := O_{k+1,\alpha} \cap h^{k+1,\beta}(M^n)$, there exists $\delta > 0$ such that the autonomous quasilinear parabolic Cauchy problem

$$\dot{v} = -Q(v)[v] + f(v),\ t > 0,\ v(0) = v_0 \tag{4.14}$$

possesses a unique maximal solution:

$$v \in C\left([0,\delta), O_{k+1,\beta}\right) \cap C\left((0,\delta), h^{k+2,\gamma}(M^n)\right).$$

The space $\mathcal{H}(Z,Y)$ was introduced in Section 3.2, where it was also shown that A being sectorial is equivalent to $-A \in \mathcal{H}(Z,Y)$. The second stated property of the solution, i.e. that $v \in C\left((0,\delta), h^{k+2,\gamma}(M^n)\right)$, is not explicitly stated in the theorem, however is mentioned in a remark at the top of page 70 of [2], also see Corollary 2.13 in [7].

Theorem 4.4.2. *For any $\rho_0 \in \tilde{V}_{1,\beta_0}$, $0 < \beta_0 < 1$, there exists $\delta > 0$ such that the PDE (1.6) with $M^n = \mathscr{C}^n_{R,d}$ and Neumann boundary condition has a unique maximal solution:*

$$\rho \in C\left([0,\delta), \tilde{V}_{1,\beta_0}\right) \cap C\left((0,\delta), h^{2,\beta_1}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)\right),$$

for any $\beta_1 \in (0, \beta_0)$. Moreover the graph over the cylinder Ω_{ρ_0} has a volume preserving mean curvature flow for $t \in [0,\delta)$, which is given, up to a tangential diffeomorphism, by $\Omega_{\rho(t)}$.

Proof. As in Section 4.3, we prove existence of solutions for the PDE (1.9) and hence obtain a solution to (1.6) with $M^n = \mathscr{C}^n_{R,d}$. We first fix $\alpha_0 \in (\beta_1, \beta_0)$ and search for a splitting $G_t(u) = -Q(u)[u] + f(u)$, $u \in V_{2,\beta_1}$, such that

$$(Q,f) \in C^{0,1}\left(V_{1,\alpha_0}, \mathcal{H}\left(h^{2,\beta_1}\left(\mathscr{T}^n_{R,d}\right), h^{0,\beta_1}\left(\mathscr{T}^n_{R,d}\right)\right) \times h^{0,\alpha_0}\left(\mathscr{T}^n_{R,d}\right)\right).$$

We use the equation for the mean curvature operator given in (2.13) to obtain the splitting $H(u) = J(u)[u] + K(u)$, where

$$J(u) := -L(u)^{-3}\left(L(u)^2 (\tilde{g}_u)^{ij} - (\tilde{g}_u)^{ik}(\tilde{g}_u)^{jl}\nabla_k u \nabla_l u\right)\frac{\partial^2}{\partial x^i \partial x^j}, \tag{4.15}$$

and

$$\begin{aligned} K(u) :=& L(u)^{-3} (\tilde{g}_u)^{ik}(\tilde{g}_u)^{jl}\left(\delta^s_i + u\mathring{h}^s_i\right)\mathring{h}_{sj}\nabla_k u \nabla_l u + L(u)^{-1}(\tilde{g}_u)^{ij}\left(\delta^s_i + u\mathring{h}^s_i\right)\mathring{h}_{sj} \\ &+ L(u)^{-3}\left(L(u)^2 (\tilde{g}_u)^{ij} - (\tilde{g}_u)^{ik}(\tilde{g}_u)^{jl}\nabla_k u \nabla_l u\right)\mathring{\Gamma}^s_{ij}\nabla_s u. \end{aligned} \tag{4.16}$$

We note that the functions are smooth on V_{1,α_0}, that is $K \in C^\infty\left(V_{1,\alpha_0}, h^{0,\alpha_0}\left(\mathscr{T}^n_{R,d}\right)\right)$ and $J \in C^\infty\left(V_{1,\alpha_0}, \mathcal{L}\left(h^{2,\beta_1}\left(\mathscr{T}^n_{R,d}\right), h^{0,\beta_1}\left(\mathscr{T}^n_{R,d}\right)\right)\right)$. We now obtain the splitting for $G_t(u)$, by defining:

$$Q(u)[v] := -L(u)\left(\fint_{\mathscr{T}^n_{R,d}} J(u)[v]\, d\mu_u - J(u)[v]\right) \tag{4.17}$$

and

$$f(u) := L(u)\left(\fint_{\mathscr{T}^n_{R,d}} K(u)\, d\mu_u - K(u)\right). \tag{4.18}$$

Note that $f \in C^\infty\left(V_{k+1,\alpha}, h^{k,\alpha}\left(\mathscr{T}^n_{R,d}\right)\right)$ for any $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$, so it only remains to show that $Q(u) \in \mathcal{H}\left(h^{2,\beta_1}\left(\mathscr{T}^n_{R,d}\right), h^{0,\beta_1}\left(\mathscr{T}^n_{R,d}\right)\right)$ for all $u \in V_{1,\alpha_0}$. We will in fact prove something more general that will be used in the subsequent corollary.

We let $k \in \mathbb{N}_0$ and $\alpha, \beta \in (0,1)$. $L(u)J(u)$ is uniformly elliptic for all $u \in V_{k+1,\alpha}$, so we use Theorem 3.2.6 to conclude $-L(u)J(u) : h^{k+2,\beta}\left(\mathscr{T}^n_{R,d}\right) \to h^{k,\beta}\left(\mathscr{T}^n_{R,d}\right)$ is sectorial. We also have the bound

$$\begin{aligned}\left\| L(u) \fint_{\mathscr{T}^n_{R,d}} J(u)[v]\, d\mu_u \right\|_{h^{k,\beta}} =& \left| \fint_{\mathscr{T}^n_{R,d}} J(u)[v]\, d\mu_u \right| \|L(u)\|_{h^{k,\beta}} \\ \le& C(u)\|v\|_{C^2} \\ \le& C(u)\|v\|_{h^{2,\epsilon}},\end{aligned}$$

for any $\epsilon \in (0,\beta)$. Therefore by the perturbation result in Proposition 3.2.7 (i) we conclude that, for all $u \in V_{k+1,\alpha}$, $-Q(u) : h^{k+2,\beta}\left(\mathscr{T}^n_{R,d}\right) \to h^{k,\beta}\left(\mathscr{T}^n_{R,d}\right)$ is sectorial, that is $Q(u) \in \mathcal{H}\left(h^{k+2,\beta}\left(\mathscr{T}^n_{R,d}\right), h^{k,\beta}\left(\mathscr{T}^n_{R,d}\right)\right)$.

Therefore we can apply Theorem 4.4.1 to obtain a solution, $u(t)$, to (1.9) such that

$$u \in C\left([0,\delta), V_{1,\beta_0}\right) \cap C\left((0,\delta), h^{2,\beta_1}\left(\mathscr{T}^n_{R,d}\right)\right), \tag{4.19}$$

and by taking $\rho(t) := u(t)|_{\mathscr{C}^n_{R,d}}$ we obtain the result. $\square$

As a corollary of this theorem we are able to obtain higher spatial regularity of $\rho(t)$ when $t > 0$. In fact we obtain that the solution is smooth instantaneously after the initial time, and hence the flow is smoothing.

Corollary 4.4.3. *Let $\rho(t)$ be the solution found in Theorem 4.4.2 with initial condition $\rho_0 \in \tilde{V}_{1,\beta_0}$, then $\rho \in C^\infty\left((0,\delta), C^\infty_{\frac{\partial}{\partial z}}\left(\mathscr{C}^n_{R,d}\right)\right) \cap C\left([0,\delta), \tilde{V}_{1,\beta_0}\right)$, i.e. for any $t \in (0,\delta)$ the hypersurface defined by $\rho(t)$ is smooth, as is the map $t \mapsto \rho(t)$.*

Proof. We again prove the regularity result by proving the same regularity result for the solution, $u(t)$, to (1.9). By the proof of Theorem 4.4.2 we have $G_t(u) = -Q(u)[u] + f(u)$, where

$$(Q,f) \in C^\infty\left(V_{k+1,\alpha}, \mathcal{H}\left(h^{k+2,\beta}\left(\mathscr{T}^n_{R,d}\right), h^{k,\beta}\left(\mathscr{T}^n_{R,d}\right)\right) \times h^{k,\alpha}\left(\mathscr{T}^n_{R,d}\right)\right), \tag{4.20}$$

for any $k \in \mathbb{N}_0$ and $\alpha, \beta \in (0,1)$. The smoothness in time then follows from the remark in the second paragraph on page 71 of [2] or from Corollary 2.13 in [7]. To get the spatial regularity we perform a bootstrapping method, similar to the proof of Theorem 1 in [19].

We will prove by induction that if $u_0 \in V_{1,\beta_0}$, then for any $k \in \mathbb{N}_0$ we have:

$$u \in C\left((0,\delta), V_{k+1,\beta_k}\right) \cap C\left((0,\delta), h^{k+2,\beta_{k+1}}\left(\mathscr{T}^n_{R,d}\right)\right), \tag{4.21}$$

where $\{\beta_j\}_{j=0}^{\infty}$ is any sequence satisfying $\beta_j \in (0, \beta_{j-1})$, we will also define the sequence $\alpha_j \in (\beta_{j+1}, \beta_j)$.

The $k = 0$ case follows from the proof of Theorem 4.4.2. We now assume (4.21) holds for some $k \in \mathbb{N}_0$ and let $\tau \in (0, \delta)$. By the inductive assumption the function $u(\tau)$ is in $V_{k+1,\beta_k} \cap h^{k+2,\beta_{k+1}}\left(\mathscr{T}^n_{R,d}\right) = V_{k+2,\beta_{k+1}}$.

Due to (4.20) we have:

$$(Q, f) \in C^{\infty}\left(V_{k+2,\alpha_{k+1}}, \mathcal{H}\left(h^{k+3,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right), h^{k+1,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right)\right) \times h^{k+1,\alpha_{k+1}}\left(\mathscr{T}^n_{R,d}\right)\right), \tag{4.22}$$

so we can apply Theorem 4.4.1 to obtain a solution to (1.6):

$$\bar{u} \in C\left([0, \bar{\delta}), V_{k+2,\beta_{k+1}}\right) \cap C\left((0, \bar{\delta}), h^{k+3,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right)\right),$$

with $\bar{u}(0) = u(\tau) \in V_{k+2,\beta_{k+1}}$.

By uniqueness of solutions to the flow we also have that $u(t) = \bar{u}(t-\tau)$ for $t \in (\tau, \tilde{\delta})$, where $\tilde{\delta} := \min(\bar{\delta} + \tau, \delta)$, and hence

$$u \in C\left([\tau, \tilde{\delta}), V_{k+2,\beta_{k+1}}\right) \cap C\left((\tau, \tilde{\delta}), h^{k+3,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right)\right).$$

We note that if $\bar{\delta} + \tau > \delta$ then $\bar{u}(t)$ extends $u(t)$ and maintains the same regularity, which contradicts the maximality of δ. Now we assume that $\bar{\delta} + \tau < \delta$. By Theorem 12.5 in [2] we conclude that either $\bar{u}(t)$ approaches the boundary of $V_{k+2,\alpha_{k+1}}$ or that $\|\bar{u}(t)\|_{h^{k+2,\theta}} \to \infty$, as $t \to \bar{\delta}$, for each $\theta \in (\alpha_{k+1}, 1)$. The same must be true of $u(t)$ as $t \to \bar{\delta} + \tau$. However, by (4.21), $u(\bar{\delta} + \tau) \in V_{k+2,\beta_{k+1}} \subset V_{k+2,\alpha_{k+1}}$, so does not tend to the boundary, and $\left\|u(\bar{\delta} + \tau)\right\|_{h^{k+2,\beta_{k+1}}} < \infty$. Since $\beta_{k+1} \in (\alpha_{k+1}, 1)$, we have a contradiction and $\bar{\delta} + \tau = \delta$, so

$$u \in C\left([\tau, \delta), V_{k+2,\beta_{k+1}}\right) \cap C\left((\tau, \delta), h^{k+3,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right)\right).$$

But this is true for all $\tau \in (0, \delta)$, hence we obtain

$$u \in C\left((0, \delta), V_{k+2,\beta_{k+1}}\right) \cap C\left((0, \delta), h^{k+3,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right)\right),$$

so by induction we have that (4.21) is true for all $k \in \mathbb{N}_0$. Therefore, for all $k \in \mathbb{N}_0$:

$$u \in C\left((0, \delta), C^{k+2}\left(\mathscr{T}^n_{R,d}\right)\right).$$

Combining this with the smoothness in time we obtain the result. $\square$

5

Stability of Weighted Volume Preserving Curvature Flows near Spheres

This chapter deals with the stability of spheres under the flow (1.4). We will again consider initial hypersurfaces that are graphs over the sphere with small height function and prove that their weighted volume preserving curvature flow exists for all time and the hypersurfaces converge to a sphere. We do this by setting up an exponentially attractive center manifold and showing that it consists entirely of spheres. Since all the results are local we will often only need to deal with the linearisation at zero, as it is the dominant term in the evolution equation. We highlight this term by rewriting the evolution equation:

$$\rho'(t) = \partial G_s(0)[\rho(t)] + \bar{G}_s(\rho(t)), \quad \bar{G}_s(v) := G_s(v) - \partial G_s(0)[v]. \tag{5.1}$$

Note that $\bar{G}_s$ is a smooth function in a neighbourhood of zero, which satisfies $\bar{G}_s(0) = 0$ and $\partial\bar{G}_s(0) = 0$. The results of this chapter, in the case of mixed volume preserving curvature flow, are included in [27].

5.1 Eigenvalues

In this section we investigate the spectrum of the operator $\partial G_s(0)$ given in equation (4.6). However, we will first consider the operator $\tilde{A}_s$ given in equation (4.11). We will denote the n-dimensional spherical harmonics of order l by $Y_{l,p}^{(n)}$ where $l \in \mathbb{N}_0$, $1 \leq p \leq M_l^{(n)}$ and

$$M_l^{(n)} := \begin{cases} \binom{l+n}{n} - \binom{l+n-2}{n} & l \geq 2, \\ \binom{l+n}{n} & l \in \{0,1\}. \end{cases} \tag{5.2}$$

Lemma 5.1.1. *The spectrum of $\tilde{A}_s : h^{2,\alpha}(\mathscr{S}_R^n) \subset h^{0,\alpha}(\mathscr{S}_R^n) \to h^{0,\alpha}(\mathscr{S}_R^n)$ is given by*

$$\sigma(\tilde{A}_s) = \left\{ -\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{(l-1)(l+n)}{R^2} : l \in \mathbb{N}_0 \right\},$$

with eigenfunctions the spherical harmonics.

Proof. Due to the compact embedding of $h^{2,\alpha}(\mathscr{S}_R^n)$ in $h^{0,\alpha}(\mathscr{S}_R^n)$ the spectrum consists entirely of eigenvalues. It is well known that the eigenfunctions of the spherical Laplacian are the spherical harmonics, $Y_{l,p}^{(n)}$, with corresponding eigenvalue $\frac{-l(l+n-1)}{R^2}$ and hence the eigenfunctions of $\tilde{A}_s$ are also the spherical harmonics and the corresponding eigenvalues are

$$\xi_l = \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\left(\frac{n}{R^2} - \frac{l(l+n-1)}{R^2}\right),$$

which proves the lemma. □

Lemma 5.1.2. *The spectrum of $\partial G_s(0) : h^{2,\alpha}(\mathscr{S}_R^n) \subset h^{0,\alpha}(\mathscr{S}_R^n) \to h^{0,\alpha}(\mathscr{S}_R^n)$ consists of a sequence of isolated non-positive eigenvalues given by:*

$$\sigma(\partial G_s(0)) = \left\{ -\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{(l-1)(l+n)}{R^2}, l \in \mathbb{N} \right\},$$

with corresponding eigenfunctions given by:

$$v_{l,p} = \begin{cases} Y_{0,1}^{(n)} & l = 1, p = 0, \\ Y_{l,p}^{(n)} & l \in \mathbb{N},\ 1 \leq p \leq M_l^{(n)}. \end{cases}$$

It follows that zero is an isolated eigenvalue of multiplicity $n+2$ and the zeroth and first order spherical harmonics form the basis for the corresponding eigenspace.

Proof. We start by noting that again the spectrum must consist solely of eigenvalues and that $Y_{0,1}^{(n)} = 1$ is an eigenfunction of $\partial G_s(0)$ with eigenvalue zero; we label this eigenfunction $v_{1,0}$. Now we note that the operator $\partial G_s(0)$ is self adjoint with respect to the L^2-inner product on $h^{2,\alpha}(\mathscr{S}_R^n)$. To see this, consider $v, w \in h^{2,\alpha}(\mathscr{S}_R^n)$ and compute:

$$\begin{aligned} \int_{\mathscr{S}_R^n} \partial G_s(0)[v] w \, d\mu_0 &= \int_{\mathscr{S}_R^n} \left(\tilde{A}_s[v] - \frac{n}{R^2} \fint_{\mathscr{S}_R^n} v \, d\mu_0 \right) w \, d\mu_0 \\ &= \int_{\mathscr{S}_R^n} \tilde{A}_s[v] w \, d\mu_0 - \frac{n}{R^2} \fint_{\mathscr{S}_R^n} v \, d\mu_0 \int_{\mathscr{S}_R^n} w \, d\mu_0 \\ &= \int_{\mathscr{S}_R^n} v \tilde{A}_s[w] \, d\mu_0 - \frac{n}{R^2} \int_{\mathscr{S}_R^n} v \, d\mu_0 \fint_{\mathscr{S}_R^n} w \, d\mu_0, \end{aligned}$$

where we use that $\tilde{A}_s$ is self adjoint with respect to the L^2-inner product, since it is a multiple of the Laplacian on the sphere plus a constant. Hence:

$$\begin{aligned}\int_{\mathscr{S}^n_R} \partial G_s(0)[v]w\, d\mu_0 &= \int_{\mathscr{S}^n_R} v\left(\tilde{A}_s[w] - \frac{n}{R^2} \fint_{\mathscr{S}^n_R} w\, d\mu_0\right) d\mu_0 \\ &= \int_{\mathscr{S}^n_R} v\partial G_s(0)[w]\, d\mu_0.\end{aligned}$$

Therefore we need only consider eigenfunctions that are L^2-orthogonal to $Y^{(n)}_{0,1} = 1$, in order to characterise the remainder of the spectrum. This means that for an eigenfunction v with eigenvalue λ we assume the property:

$$\int_{\mathscr{S}^n_R} v\, d\mu_0 = 0,$$

and hence

$$\lambda v = \partial G_s(0)[v] = \tilde{A}_s[v].$$

Thus the remaining eigenfunctions of $\partial G_s(0)$ are precisely the remaining eigenfunctions of $\tilde{A}_s$, which are given in Lemma 5.1.1. □

5.2 Center Manifold

This section deals with the fact that having a nontrivial nullspace of $\partial G_s(0)$ means that we are unable to obtain a priori bounds on the solution. To address this we shall construct a local invariant center manifold for the flow (5.1) and investigate its contents.

We start the investigation by providing an existence theorem for center manifolds along with some properties. We let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and

$$A : h^{k+2,\alpha_0}(M^n) \to h^{k,\alpha_0}(M^n)$$

be a sectorial operator for some $\alpha_0 \in (0,\alpha)$. Assume that $\sigma_+(A)$ consists of a finite number of isolated eigenvalues and define:

$$X^c := P_+\left(h^{k+2,\alpha}(M^n)\right), \quad X^s_{k,\alpha} := (I - P_+)\left(h^{k,\alpha}(M^n)\right),$$

the *center subspace* and *stable subspace* respectively. We also note that since X^c is finite dimensional all norms on it are equivalent and we don't include any subscripts.

Theorem 5.2.1 (Theorem 9.2.2 [38]). *Let $G \in C^1\left(O, h^{k,\alpha}(M^n)\right)$ with $G(0) = 0$ and $\partial G(0) = 0$, where $O \subset h^{k+2,\alpha}(M^n)$ is a neighbourhood of zero. There exists $R_1 > 0$ such that for any $r \in (0, R_1]$ there is a Lipschitz continuous function $\gamma_r : X^c \to X^s_{k+2,\alpha}$ such that the graph of γ_r is an invariant for the system:*

$$x'(t) = A_+[x(t)] + P_+\left[\tilde{G}_r\left(x(t), y(t)\right)\right], \quad y'(t) = A_-[y(t)] + (I - P_+)\left[\tilde{G}_r\left(x(t), y(t)\right)\right], \tag{5.3}$$

$$x(0) = x_0 \in X^c, \ y(0) = y_0 \in X^s_{k+2,\alpha},$$

where $\tilde{G}_r(x, y) := G\left(\eta\left(\frac{x}{r}\right) x + y\right)$ and $\eta : X^c \to \mathbb{R}$ is a cut-off function such that

$$0 \le \eta(x) \le 1, \ \eta(x) = 1 \text{ if } \|x\|_{h^{k,\alpha}} \le 1, \ \eta(x) = 0 \text{ if } \|x\|_{h^{k,\alpha}} \ge 2.$$

Furthermore γ_r is the unique map satisfying

$$\gamma_r(x) = \int_{-\infty}^0 e^{-sA_-}(I - P_+)\left[\tilde{G}_r\left(w_r(s; x, \gamma_r), \gamma_r(w_r(s; x, \gamma_r))\right)\right] ds, \tag{5.4}$$

where $w_r(s; x, \gamma_r)$ is the solution to

$$w'(s) = A_+[w(s)] + P_+\left[\tilde{G}_r(w(s), \gamma_r(w(s)))\right], \ w(0) = x. \tag{5.5}$$

If in addition G is l times continuously differentiable, with $l \ge 2$, then there exists $R_l > 0$ such that if $r \in (0, R_l]$ then $\gamma_r \in C^{l-1,1}$ and

$$\partial\gamma_r(x)\left[A_+[x] + P_+\left[\tilde{G}_r\left(x, \gamma_r(x)\right)\right]\right] = A_-[\gamma_r(x)] + (I - P_+)\left[\tilde{G}_r\left(x, \gamma_r(x)\right)\right].$$

Proposition 5.2.2 (Proposition 9.2.3 [38]). *Let the assumptions in Theorem 5.2.1 hold with G at least twice continuously differentiable. There exists $\tilde{R}_2 > 0$ such that if $r \in (0, \tilde{R}_2]$ and $(x_r(t), y_r(t)) \in X^c \times B_{X^s_{k+2,\alpha}, r}(0)$ is a solution to (5.3) for all $t \ge 0$ then*

$$\|y_r(t) - \gamma_r(x_r(t))\|_{h^{k+2,\alpha}} \le M(\omega)e^{-\omega t}\|y_0 - \gamma_r(x_0)\|_{h^{k+2,\alpha}}, \tag{5.6}$$

for any $\omega \in (0, \omega_-)$, see (1.18). Further if $\|x_0\|_{h^{k,\alpha}}$ and $\|y_0\|_{h^{k+2,\alpha}}$ are small enough then the solution to (5.3) satisfies the assumptions.

Note that [38] starts by assuming that $\|x_0\|_{h^{k,\alpha}}$ and $\|y_0\|_{h^{k+2,\alpha}}$ are small before deriving the estimate (5.6). However, it is clear from the proof that once long time existence is obtained, this assumption is not needed. Stating the proposition in this manner also allows us to prove, by taking $t \to \infty$ in (5.6), the following corollary:

Corollary 5.2.3. *Let $r \in (0, \tilde{R}_2]$ and suppose $(x_r(t), y_r(t)) \in X^c \times B_{X^s_{k+2,\alpha}, r}(0)$ is a stationary solution to (5.3), i.e. $(x_r(t), y_r(t)) = (x_0, y_0)$ for all $t \ge 0$. Then $y_0 = \gamma_r(x_0)$.*

This result is a special case of Theorem 2.3 in [44], where it was proved that any eternal bounded solution to (5.7) must be contained in the center manifold. However, the graph function used in [44] is defined differently to above and while they can be seen to be equivalent on $B_{X^c,r}(0)$, the above corollary is enough for our purposes.

In the case of weighted volume preserving curvature flows for graphs over spheres, the local system we consider is:

$$\begin{aligned} x'(t) &= \partial G_s(0)_+[x(t)] + P_+\left[\bar{G}_s\left(\eta\left(\tfrac{x(t)}{r}\right) + y(t)\right)\right], & x(0) &= P_+[\rho_0], \\ y'(t) &= \partial G_s(0)_-[y(t)] + (I - P_+)\left[\bar{G}_s\left(\eta\left(\tfrac{x(t)}{r}\right) + y(t)\right)\right], & y(0) &= (I - P_+)[\rho_0], \end{aligned} \tag{5.7}$$

for $\rho_0 \in h^{2,\alpha}\left(\mathscr{S}^n_R\right)$.

Theorem 5.2.4. *There exists $\tilde{R}_2 > 0$ such that for any $r \in (0, \tilde{R}_2]$ there is a function $\gamma_r \in C^{1,1}\left(X^c, X^s_{2,\alpha}\right)$ such that $\gamma_r(0) = 0$ and $\partial\gamma_r(0) = 0$. Further, $\mathcal{M}^c_r :=$ graph(γ_r) has dimension $n+2$ and if $\rho_0 \in \mathcal{M}^c_r$, then the solution to (5.1), $\rho(t)$, is in $\mathcal{M}^c_r$ as long as $P_+[\rho(t)] \in B_{X^c,r}(0)$.*

We call $\mathcal{M}^c_r$ a *locally invariant* manifold. Note that since $\partial G_s(0)$ is self adjoint with respect to the L^2-inner product, $\langle \cdot, \cdot \rangle$, it commutes with the L^2-orthogonal projection onto X^c:

$$P[\rho] := \sum_{a=0}^{n+1} \frac{\langle \rho, v_{1,a} \rangle}{\langle v_{1,a}, v_{1,a} \rangle} v_{1,a}. \tag{5.8}$$

That is, $P\left[\partial G_s(0)[v]\right] = \partial G_s(0)\left[P[v]\right] = 0$ for all $v \in h^{2,\alpha}\left(\mathscr{S}^n_R\right)$. Notably this means that $P\left[h^{k+2,\alpha}\left(\mathscr{S}^n_R\right)\right] = N\left(\partial G_s(0)\right)$ and $(I-P)\left[h^{k+2,\alpha}\left(\mathscr{S}^n_R\right)\right] = \mathrm{Range}\left(\partial G_s(0)\right)$, so $P = P_+$, the spectral projection associated to $\sigma_+(\partial G_s(0))$. This also means that $\partial G_s(0)_+ = 0$. Note that if we define $a_{k,\alpha} := \sum_{a=0}^{n+1} \frac{\|v_{1,a}\|_{h^{k,\alpha}} \int_{\mathscr{S}^n_R} |v_{1,a}|\, d\mu_0}{\langle v_{1,a}, v_{1,a} \rangle}$ we have that for any $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$:

$$\|P_+[\rho]\|_{h^{k,\alpha}} \le a_{k,\alpha} \|\rho\|_{C^0}, \quad \|(I-P_+)[\rho]\|_{h^{k,\alpha}} \le (1 + a_{k,\alpha})\|\rho\|_{h^{k,\alpha}}. \tag{5.9}$$

We now set

$$\mathcal{S} := \left\{\rho \in U_{2,\alpha} : \Omega_\rho \text{ is a sphere}\right\}.$$

Lemma 5.2.5. *There exists a neighbourhood of zero, $W_s \subset X^c$, such that $\mathcal{M}^c_r$ and $\mathcal{S}$ are identical inside $(W_s \cap B_{X^c,r}(0)) \times B_{X^s_{2,\alpha},r}(0)$, for any $r \in (0, \tilde{R}_2]$.*

Proof. Firstly, since any $\rho_0 \in \mathcal{S} \cap \left(B_{X^c,r}(0) \times B_{X^s_{2,\alpha},r}(0)\right)$ is a stationary solution to (5.1) and hence also to (5.7), we use Corollary 5.2.3 to conclude that $\rho_0 \in \mathcal{M}^c_r$. The rest of the proof follows as in [21]: If $\rho \in \mathcal{S}$, then we obtain the parameters $\boldsymbol{y} = (y_0, \dots, y_{n+1}) \in \mathbb{R}^{n+2}$, where $y_0 := R' - R$, R' is the radius of Ω_ρ, and $(y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$ is the center of the graph. Since

$$\boldsymbol{X}_\rho = R\left(Y^{(n)}_{1,1}, \dots, Y^{(n)}_{1,n+1}\right) + \rho\left(Y^{(n)}_{1,1}, \dots, Y^{(n)}_{1,n+1}\right)$$

we have the relationship:

$$(R + y_0)^2 = R'^2 = \sum_{a=1}^{n+1}\left((R+\rho)Y^{(n)}_{1,a} - y_a\right)^2 = (R+\rho)^2 - 2(R+\rho)\sum_{a=1}^{n+1} y_a Y^{(n)}_{1,a} + \sum_{a=1}^{n+1} y_a^2. \tag{5.10}$$

Solving this equation for $R + \rho$ gives

$$R + \rho = \sum_{a=1}^{n+1} y_a Y^{(n)}_{1,a} + \sqrt{\left(\sum_{a=1}^{n+1} y_a Y^{(n)}_{1,a}\right)^2 + (R+y_0)^2 - \sum_{a=1}^{n+1} y_a^2},$$

and by setting

$$\chi(\boldsymbol{y}) := \sum_{a=1}^{n+1} y_a v_{1,a} - R v_{1,0} + \sqrt{\left(\sum_{a=1}^{n+1} y_a v_{1,a}\right)^2 + \left((R+y_0)^2 - \sum_{a=1}^{n+1} y_a^2\right) v_{1,0}}, \tag{5.11}$$

we have $\rho = \chi(\boldsymbol{y})$. We will consider $\chi : U \subset \mathbb{R}^{n+2} \to h^{2,\alpha}(\mathscr{S}^n_R)$, where U is a neighbourhood of zero such that χ is smooth on it. It is clear from the construction that for any $\rho \in \mathcal{S}$, with sufficiently small norm, there exists a $\boldsymbol{y} \in U$ such that $\rho = \chi(\boldsymbol{y})$. We now calculate the linearisation of χ at zero acting on $\boldsymbol{x} \in \mathbb{R}^{n+2}$:

$$\begin{aligned}
\partial\chi(0)[\boldsymbol{x}] &= \sum_{a=0}^{n+1} \frac{\partial \chi}{\partial y_a}(0) x_a \\
&= \left.\frac{(R+y_0)v_{1,0}}{\sqrt{\left(\sum_{a=1}^{n+1} y_a v_{1,a}\right)^2 + \left((R+y_0)^2 - \sum_{a=1}^{n+1} y_a^2\right) v_{1,0}}}\right|_{\boldsymbol{y}=0} x_0 \\
&\quad + \sum_{a=1}^{n+1}\left.\left(v_{1,a} + \frac{v_{1,a}\sum_{b=1}^{n+1} y_b v_{1,b} - y_a v_{1,0}}{\sqrt{\left(\sum_{b=1}^{n+1} y_b v_{1,b}\right)^2 + \left((R+y_0)^2 - \sum_{b=1}^{n+1} y_b^2\right) v_{1,0}}}\right)\right|_{\boldsymbol{y}=0} x_a \\
&= \sum_{a=0}^{n+1} x_a v_{1,a}.
\end{aligned} \tag{5.12}$$

We now consider the map $\bar{\chi}(\boldsymbol{y}) : U \to X^c$ given by $\bar{\chi}(\boldsymbol{y}) := P_+[\chi(\boldsymbol{y})]$. Again the linearisation at zero is given by $\partial\bar{\chi}(0)[\boldsymbol{x}] = \sum_{a=0}^{n+1} x_a v_{1,a}$ and hence is the identity map with respect to the basis $v_{1,p}$, $0 \leq p \leq n+1$, of X^c. Therefore there exists a neighbourhood of zero, $V \subset X^c$, such that $\bar{\chi}$ is a diffeomorphism from V onto its image, $W_s \subset X^c$. Further, the function $\bar{\gamma}_s := \chi \circ \bar{\chi}^{-1} - I : W_s \to X^s_{2,\alpha}$ parametrises $\mathcal{S}$ as a graph over X^c locally. Since from the first remark of the proof we have that $\mathcal{S} \cap \left(B_{X^c,r}(0) \times B_{X^s_{2,\alpha},r}(0)\right) \subset \mathcal{M}^c_r$, we conclude that $\mathcal{S}$ and $\mathcal{M}^c_r$ coincide inside $(W_s \cap B_{X^c,r}(0)) \times B_{X^s_{2,\alpha},r}(0)$. Note also that $\bar{\gamma}_s|_{W_s \cap B_{X^c,r}(0)} = \gamma_r|_{W_s \cap B_{X^c,r}(0)}$. □

5.3 Convergence to a Sphere

In this section we prove the main result of the chapter, that the spheres are stable under the weighted volume preserving curvature flows. The main result we will be using is again from [38]:

Proposition 5.3.1 (Proposition 9.2.4 [38])**.** *Let the assumptions of Theorem 5.2.2 hold and x_0, y_0 satisfy the same smallness condition. If $r \in (0, \tilde{R}_2]$, there exists $\bar{x} \in X^c$ such that the system (5.3) has a solution for all $t \geq 0$ and*

$$\|x_r(t) - w_r(t; \bar{x}, \gamma_r)\|_{h^{k,\alpha}} + \|y_r(t) - \gamma_r(w_r(t; \bar{x}, \gamma_r))\|_{h^{k+2,\alpha}} \leq C(\omega)e^{-\omega t}\|y_0 - \gamma_r(x_0)\|_{h^{k+2,\alpha}},$$

for any $\omega \in (0, \omega_-)$.

Theorem 5.3.2. *There exists a neighbourhood of zero, $O_s \subset h^{2,\alpha}(\mathscr{S}^n_R)$, such that for $\rho_0 \in O_s$, then the flow (1.4) with initial hypersurface Ω_{ρ_0} exists for all time. Furthermore, the hypersurfaces converge exponentially fast to a sphere as $t \to \infty$, with respect to the $h^{2,\alpha}(\mathscr{S}^n_R)$ topology, $\alpha \in (0, 1)$.*

Proof. We fix $r \in (0, \tilde{R}_2]$ and start the proof by noting that if $x \in W_s \cap B_{X^c,r}(0)$ then $x + \gamma_r(x)$ defines a sphere by Lemma 5.2.5 and hence is a stationary solution to equation (5.1), i.e.

$$\begin{aligned} 0 &= \partial G_s(0)[x + \gamma_r(x)] + \bar{G}_s(x + \gamma_r(x)) \\ &= \partial G_s(0)[x + \gamma_r(x)] + \bar{G}_s\left(\eta\left(\frac{x}{r}\right)x + \gamma_r(x)\right). \end{aligned}$$

By taking the projection of this equation we see that $x = P_+[x + \gamma_r(x)]$ is a stationary solution of

$$w'(t) = P_+\left[\bar{G}_s\left(\eta\left(\frac{w(t)}{r}\right)w(t) + \gamma_r(w(t))\right)\right],\ w(0) = x,$$

hence $w_r(t; x, \gamma_r) = x$ for all $x \in W_s \cap B_{X^c,r}(0)$.

We now consider ρ_0 small enough so that, by equation (5.9), we can apply Proposition 5.3.1, with $\omega_- = \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n+2}{R^2}$, to obtain, for any $\omega \in (0, \omega_-)$, $\bar{x} \in X^c$ such that:

$$\begin{aligned}\|x_r(t) - w_r(t; \bar{x}, \gamma_r)\|_{h^{0,\alpha}} + &\|y_r(t) - \gamma_r(w_r(t; \bar{x}, \gamma_r))\|_{h^{2,\alpha}} \\ &\leq Ce^{-\omega t}\|(I - P_+)[\rho_0] - \gamma_r(P_+[\rho_0])\|_{h^{2,\alpha}},\end{aligned} \tag{5.13}$$

where $(x_r(t), y_r(t))$ solves (5.7). By evaluating this at $t = 0$ we obtain the bound:

$$\|P_+[\rho_0] - \bar{x}\|_{h^{0,\alpha}} \leq C \|(I - P_+)[\rho_0] - \gamma_r (P_+[\rho_0])\|_{h^{2,\alpha}}. \tag{5.14}$$

This allows us to bound $\bar{x}$ in terms of ρ_0.

$$\begin{aligned}\|\bar{x}\|_{h^{0,\alpha}} &\leq \|P_+[\rho_0] - \bar{x}\|_{h^{0,\alpha}} + \|P_+[\rho_0]\|_{h^{0,\alpha}} \\ &\leq C \|(I - P_+)[\rho_0] - \gamma_r (P_+[\rho_0])\|_{h^{2,\alpha}} + a_{0,\alpha}\|\rho_0\|_{h^{2,\alpha}} \\ &\leq C (\|(I - P_+)[\rho_0]\|_{h^{2,\alpha}} + \|\gamma_r (P_+[\rho_0])\|_{h^{2,\alpha}}) + a_{0,\alpha}\|\rho_0\|_{h^{2,\alpha}} \\ &\leq C ((1 + a_{2,\alpha}) \|\rho_0\|_{h^{2,\alpha}} + b_r \|P_+[\rho_0]\|_{h^{0,\alpha}}) + a_{0,\alpha}\|\rho_0\|_{h^{2,\alpha}} \\ &\leq C\|\rho_0\|_{h^{2,\alpha}},\end{aligned} \tag{5.15}$$

where we use (5.9) and where b_r is the Lipschitz constant of γ_r. Therefore if ρ_0 is small enough we have that $\bar{x} \in W_s \cap B_{X^c,r}(0)$ and hence, by the first part of the proof, $w_r(t; \bar{x}, \gamma_r) = \bar{x}$. Equation (5.13) now simplifies to:

$$\|x_r(t) - \bar{x}\|_{h^{0,\alpha}} + \|y_r(t) - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \leq Ce^{-\omega t} \|(I - P_+)[\rho_0] - \gamma_r (P_+[\rho_0])\|_{h^{2,\alpha}}. \tag{5.16}$$

The last part of the proof involves proving that $x_r(t) \in W_s \cap B_{X^c,r}(0)$ for all $t \geq 0$ and hence $\rho(t) = x_r(t) + y_r(t)$ is a solution to (5.1) for all $t \geq 0$. We use a similar calculation to the one we used to derive (5.15):

$$\begin{aligned}\|x_r(t)\|_{h^{0,\alpha}} &\leq \|x_r(t) - \bar{x}\|_{h^{0,\alpha}} + \|\bar{x}\|_{h^{0,\alpha}} \\ &\leq Ce^{-\omega t} \|(I - P_+) [\rho_0] - \gamma_r (P_+[\rho_0])\|_{h^{2,\alpha}} + C\|\rho_0\|_{h^{2,\alpha}} \\ &\leq C (\|(I - P_+) [\rho_0]\|_{h^{2,\alpha}} + \|\gamma_r (P_+[\rho_0])\|_{h^{2,\alpha}}) + C\|\rho_0\|_{h^{2,\alpha}} \\ &\leq C ((1 + a_{2,\alpha}) \|\rho_0\|_{h^{2,\alpha}} + b_r \|P_+[\rho_0]\|_{h^{0,\alpha}}) + C\|\rho_0\|_{h^{2,\alpha}} \\ &\leq C\|\rho_0\|_{h^{2,\alpha}}.\end{aligned} \tag{5.17}$$

Therefore by considering ρ_0 small enough we have that $\rho(t) = x_r(t) + y_r(t)$ is a solution to (5.1) for all $t \geq 0$ with $P_+[\rho(t)] = x_r(t)$ and $(I - P_+)[\rho(t)] = y_r(t)$. Hence:

$$\begin{aligned}
\|\rho(t) - (\bar{x} + \gamma_r(\bar{x}))\|_{h^{2,\alpha}} &= \|P_+[\rho(t)] - \bar{x} + (I - P_+)[\rho(t)] - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \\
&\leq \|P_+[\rho(t)] - \bar{x}\|_{h^{2,\alpha}} + \|(I - P_+)[\rho(t)] - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \\
&\leq C\|P_+[\rho(t)] - \bar{x}\|_{h^{0,\alpha}} + \|(I - P_+)[\rho(t)] - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \\
&\leq Ce^{-\omega t} \|(I - P_+)[\rho_0] - \gamma_r(P_+[\rho_0])\|_{h^{2,\alpha}},
\end{aligned}$$

where we used equivalence of norms on X^c. We have therefore found that $\rho(t)$ converges exponentially to $\bar{x} + \gamma_r(\bar{x})$, which by Lemma 5.2.5 is a sphere. □

The previous theorem proves that the spheres are stable stationary solutions to the weighted volume preserving curvature flow, that is hypersurfaces close to a sphere under the flow converge to a sphere near the original one. We also have the following corollary concerning the stability of hypersurfaces that converge to spheres under the flow. We find that hypersurfaces near them also converge to spheres.

Corollary 5.3.3. *Let $\rho(t)$ be a solution to the equation (1.5), which exists for all time and converges to zero. Suppose further that $\frac{\partial F}{\partial \kappa_i}\left(\boldsymbol{\kappa}_{\rho(t)}\right) > 0$ for all $t \in [0, \infty)$ and $i = 1, \ldots, n$. Then there exists a neighbourhood, $O_{s,4}$, of $\rho(0)$ in $h^{2,\alpha}(\mathscr{S}^n_R)$, $0 < \alpha < 1$, such that for every $v_0 \in O_{s,4}$ the solution to (1.5) with initial condition v_0 exists for all time and converges to a function near zero whose graph is a sphere.*

Proof. This follows by the same arguments given in [26]. Since $\rho(t)$ converges to zero in the $h^{2,\alpha}$-topology, there exists a time, T, such that $\rho(T) \in O_s$ (given in Theorem 5.3.2) and, as O_s is open, there exists an open ball, $B_{h^{2,\alpha}(\mathscr{S}^n_R),\epsilon}(\rho(T)) \subset O_s$, of radius ϵ centred at $\rho(T)$. We consider the linearisation of $h(\rho)$ about $\rho(t)$:

$$\begin{aligned}
\partial h(\rho(t))[v] =& \frac{1}{\int_{\mathscr{S}^n_R} \Xi\left(\boldsymbol{\kappa}_{\rho(t)}\right) d\mu_{\rho(t)}} \int_{\mathscr{S}^n_R} \partial\left(F(\boldsymbol{\kappa}_\rho)\,\Xi(\boldsymbol{\kappa}_\rho)\,\mu(\rho)\right)\big|_{\rho=\rho(t)} [v]\, d\mu_0 \\
& - \frac{\int_{\mathscr{S}^n_R} F\left(\boldsymbol{\kappa}_{\rho(t)}\right) \Xi\left(\boldsymbol{\kappa}_{\rho(t)}\right) d\mu_{\rho(t)}}{\left(\int_{\mathscr{S}^n_R} \Xi\left(\boldsymbol{\kappa}_{\rho(t)}\right) d\mu_{\rho(t)}\right)^2} \int_{\mathscr{S}^n_R} \partial\left(\Xi(\boldsymbol{\kappa}_\rho)\,\mu(\rho)\right)\big|_{\rho=\rho(t)} d\mu_0 \\
=& \frac{1}{\int_{\mathscr{S}^n_R} \Xi\left(\boldsymbol{\kappa}_{\rho(t)}\right) d\mu_{\rho(t)}} \Bigg(\int_{\mathscr{S}^n_R} \partial F(\boldsymbol{\kappa}_\rho)\big|_{\rho=\rho(t)} [v] \Xi\left(\boldsymbol{\kappa}_{\rho(t)}\right) d\mu_{\rho(t)} \\
& \qquad + \int_{\mathscr{S}^n_R} F\left(\boldsymbol{\kappa}_{\rho(t)}\right) \partial\left(\Xi(\boldsymbol{\kappa}_\rho)\,\mu(\rho)\right)\big|_{\rho=\rho(t)} [v]\, d\mu_0 \\
& \qquad - h(\rho(t)) \int_{\mathscr{S}^n_R} \partial\left(\Xi(\boldsymbol{\kappa}_\rho)\,\mu(\rho)\right)\big|_{\rho=\rho(t)} [v]\, d\mu_0 \Bigg).
\end{aligned}$$

Therefore the linearisation of $G_s(\rho)$ around $\rho(t)$ is:

$$\partial G_s(\rho(t)) = A_{\rho(t)}[v] + \frac{L(\rho(t))}{\int_{\mathscr{S}^n_R} \Xi\left(\boldsymbol{\kappa}_{\rho(t)}\right) d\mu_{\rho(t)}} \int_{\mathscr{S}^n_R} B_{\rho(t)}[v]\, d\mu_{\rho(t)},$$

where

$$A_\rho[v] := (h(\rho) - F(\boldsymbol{\kappa}_\rho))\, \partial L(\rho)[v] - L(\rho)\partial F(\boldsymbol{\kappa}_\rho)[v],$$

and

$$B_\rho[v] := \Xi(\boldsymbol{\kappa}_\rho)\, \partial F(\boldsymbol{\kappa}_\rho)[v] - (h(\rho) - F(\boldsymbol{\kappa}_\rho))\left(\partial \Xi(\boldsymbol{\kappa}_\rho)[v] + \Xi(\boldsymbol{\kappa}_\rho)\, \partial \ln|\mu(\rho)[v]|\right).$$

The fact that $L(\rho) > 0$ and is a first order operator, together with the condition that $\frac{\partial F}{\partial \kappa_i}\left(\boldsymbol{\kappa}_{\rho(t)}\right) > 0$ for all $t \in [0, \infty)$ and $i = 1, \ldots, n$, ensures that the operator $-A_{\rho(t)}$ is uniformly elliptic for every $t \in [0, \infty)$ (see [4]). Hence by Theorem 3.2.6, $A_{\rho(t)} : h^{2,\alpha_0}(\mathscr{S}^n_R) \to h^{0,\alpha_0}(\mathscr{S}^n_R)$ is sectorial.

We also have that $B_\rho : C^2(\mathscr{S}^n_R) \to C^0(\mathscr{S}^n_R)$ is a bounded second order operator and therefore the global term in the linearisation is in $\mathcal{L}\left(h^{2,\beta}(\mathscr{S}^n_R), h^{0,\alpha_0}(\mathscr{S}^n_R)\right)$, for any $\beta \in (0,1)$. By choosing $\beta < \alpha_0$ we can apply the perturbation result in Proposition 3.2.7 (i) to conclude that $\partial G_s(\rho(t))$ is sectorial for all $t \in [0, T]$. Hence, by Proposition 3.2.8, $\partial G_s(\rho)$ is sectorial for all $\rho \in O(\rho(t)) \subset h^{2,\alpha_0}(\mathscr{S}^n_R)$, a neighbourhood of $\rho(t)$.

By Theorem 8.4.4 in [38] the flow depends continuously on the initial condition in a neighbourhood of ρ_0. Therefore there exists a ball $B_{h^{2,\alpha}\left(\mathscr{S}^n_R\right),\delta}(\rho_0)$ such that if $v_0 \in B_{h^{2,\alpha}\left(\mathscr{S}^n_R\right),\delta}(\rho_0)$ then the solution, $v(t)$, to (1.5) with initial condition v_0 exists for $t \in [0, T]$ and $v(T) \in B_{h^{2,\alpha}\left(\mathscr{S}^n_R\right),\epsilon}(\rho(T))$. Since $v(T)$ is in O_s, by Theorem 5.3.2 the solution to (1.5) with initial condition $v(T)$ converges to a function near zero that defines a sphere. By uniqueness of the flow we get the result. $\square$

6

Stability of Weighted Volume Preserving Curvature Flows near Finite Cylinders

In this chapter we look at the stability of finite cylinders under the flow (1.4), with the boundary condition that the hypersurfaces meet a pair of parallel hyperplanes orthogonally. We will consider initial hypersurfaces that are graphs over cylinders of length d and radius R. When the height function is small, we prove that if the radius of the cylinder satisfies a certain condition, then its weighted volume preserving curvature flow exists for all time and the hypersurfaces converge to a cylinder. To deal with the boundary conditions we will continue to work with the PDE (1.7), then translate the results to the geometric setting. This chapter follows the same pattern as Chapter 5 where we set up an exponentially attractive center manifold and prove it consists entirely of cylinders. We will again rewrite the PDE to highlight the dominant term:

$$u'(t) = \partial G_t(0)[u(t)] + \bar{G}_t(u(t)), \tag{6.1}$$

where

$$\bar{G}_t(v) := G_t(v) - \partial G_t(0)[v].$$

Note that $\bar{G}_t$ is a smooth function in a neighbourhood of zero, which satisfies $\bar{G}_t(0) = 0$ and $\partial\bar{G}_t(0) = 0$.

6.1 Eigenvalues

In this section we investigate the spectrum of the operator $\partial G_t(0)$ given in equation (4.8). However, we will first consider the operator $\tilde{A}_t$ given in equation (4.13).

Lemma 6.1.1. *The spectrum of* $\tilde{A}_t : h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right) \subset h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right) \to h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ *is given by*

$$\sigma(\tilde{A}_t) = \left\{ -\left(\frac{\frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)\, m^2\pi^2}{d^2} + \frac{\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\,(l-1)(l+n-1)}{R^2} \right) : m, l \in \mathbb{N}_0 \right\},$$

with eigenfunctions:

$$v_{l,p,m,1}(\boldsymbol{q}, z) = \cos\left(\frac{m\pi z}{d}\right) Y^{(n-1)}_{l,p}(\boldsymbol{q}),\ 1 \le p \le M^{(n-1)}_l,$$

$$v_{l,p,m,2}(\boldsymbol{q}, z) = \sin\left(\frac{m\pi z}{d}\right) Y^{(n-1)}_{l,p}(\boldsymbol{q}),\ 1 \le p \le M^{(n-1)}_l,$$

where the constants $M^{(n-1)}_l$ *and the spherical harmonics* $Y^{(n-1)}_{l,p}$ *are defined at the start of Section 5.1.*

Proof. For ease of notation we set $F_{1,0} = \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)$ and $F_{n,0} = \frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)$. Due to the compact embedding of $h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ in $h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ the spectrum consists entirely of eigenvalues. The operator $\tilde{A}_t$ is also self adjoint with respect to the L^2-inner product on $h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$. To see this we consider $v, w \in h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$:

$$\begin{aligned}
\int_{\mathscr{T}^n_{R,d}} \tilde{A}_t[v] w \, d\mu_0 =& \int_{\mathscr{T}^n_{R,d}} \left(F_{1,0}\Delta_{\mathscr{S}^{n-1}_R} v + F_{1,0}\frac{(n-1)}{R^2} v + F_{n,0}\frac{\partial^2 v}{\partial z^2} \right) w \, d\mu_0 \\
=& F_{1,0} \int_{\mathscr{S}^1_{\frac{d}{\pi}}} \int_{\mathscr{S}^{n-1}_R} w \Delta_{\mathscr{S}^{n-1}_R} v R^{n-1} \, d\sigma \, dz + F_{1,0}\frac{(n-1)}{R^2} \int_{\mathscr{T}^n_{R,d}} vw \, d\mu_0 \\
&+ F_{n,0} \int_{\mathscr{S}^{n-1}_R} \int_{\mathscr{S}^1_{\frac{d}{\pi}}} w \frac{\partial^2 v}{\partial z^2} R^{n-1} \, dz \, d\sigma \\
=& F_{1,0} \int_{\mathscr{S}^1_{\frac{d}{\pi}}} \int_{\mathscr{S}^{n-1}_R} v \Delta_{\mathscr{S}^{n-1}_R} w R^{n-1} \, d\sigma \, dz + F_{1,0}\frac{(n-1)}{R^2} \int_{\mathscr{T}^n_{R,d}} vw \, d\mu_0 \\
&+ F_{n,0} \int_{\mathscr{S}^{n-1}_R} \int_{\mathscr{S}^1_{\frac{d}{\pi}}} v \frac{\partial^2 w}{\partial z^2} R^{n-1} \, dz \, d\sigma \\
=& \int_{\mathscr{T}^n_{R,d}} v \tilde{A}_t[w] \, d\mu_0,
\end{aligned}$$

where $d\sigma$ is the volume form on $\mathscr{S}^{n-1}_1$

To further analyse the spectrum of $\tilde{A}_t$ we consider eigenfunctions that have the factorisation $v(\boldsymbol{q}, z) = X(\boldsymbol{q})Z(z)$, where $\boldsymbol{q} \in \mathscr{S}_R^{n-1}$ and $z \in \mathscr{S}^1_{\frac{d}{\pi}}$, $-d < z \leq d$. Therefore

$$\left(F_{1,0}\left(\Delta_{\mathscr{S}_R^{n-1}} + \frac{n-1}{R^2}\right) + F_{n,0}\frac{\partial^2}{\partial z^2}\right) X(\boldsymbol{q})Z(z) = \lambda X(\boldsymbol{q})Z(z),$$

so after expanding the terms we have

$$F_{1,0}Z(z)\Delta_{\mathscr{S}_R^{n-1}}X(\boldsymbol{q}) + F_{n,0}X(\boldsymbol{q})Z''(z) + \left(F_{1,0}\frac{n-1}{R^2} - \lambda\right) X(\boldsymbol{q})Z(z) = 0.$$

Therefore we can separate the variables:

$$-\frac{\Delta_{\mathscr{S}_R^{n-1}}X(\boldsymbol{q})}{X(\boldsymbol{q})} = \frac{F_{n,0}Z''(z)}{F_{1,0}Z(z)} + \left(\frac{n-1}{R^2} - \frac{\lambda}{F_{1,0}}\right).$$

We set both sides to equal a constant $\xi \in \mathbb{R}$, which gives

$$\Delta_{\mathscr{S}_R^{n-1}}X(\boldsymbol{q}) = -\xi X(\boldsymbol{q}),\ Z''(z) = \frac{1}{F_{n,0}}\left(F_{1,0}\xi - F_{1,0}\frac{(n-1)}{R^2} + \lambda\right) Z(z). \tag{6.2}$$

As given in Section 5.1 the eigenfunctions of the spherical Laplacian are the spherical harmonics, so $X_{l,p}(\boldsymbol{q}) = Y_{l,p}^{(n-1)}(\boldsymbol{q})$ for $1 \leq p \leq M_l^{(n-1)}$, and the corresponding eigenvalues are $-\xi_l = \frac{-l(l+n-2)}{R^2}$. Substituting ξ_l into the second equation in (6.2) gives:

$$Z''(z) = \frac{1}{F_{n,0}}\left(F_{1,0}\frac{(l-1)(l+n-1)}{R^2} + \lambda\right) Z(z).$$

The eigenfunctions of this system are again the spherical harmonics, but this time in one-dimension. They can be written as $Z_{m,1}(z) = \cos\left(\frac{m\pi z}{d}\right)$, for $m \in \mathbb{N}_0$, and $Z_{m,2}(z) = \sin\left(\frac{m\pi z}{d}\right)$, for $m \in \mathbb{N}$. Hence we have the relationship

$$\frac{1}{F_{n,0}}\left(F_{1,0}\frac{(l-1)(l+n-1)}{R^2} + \lambda\right) = -\frac{m^2\pi^2}{d^2}.$$

Therefore the eigenvalues of $\tilde{A}_t$ are:

$$\lambda_{l,m} = -\left(F_{n,0}\frac{m^2\pi^2}{d^2} + F_{1,0}\frac{(l-1)(l+n-1)}{R^2}\right),$$

with corresponding eigenfunctions:

$$v_{l,p,m,1}(\boldsymbol{q}, z) = \cos\left(\frac{m\pi z}{d}\right) Y_{l,p}^{(n-1)}(\boldsymbol{q}),\ v_{l,p,m,2}(\boldsymbol{q}, z) = \sin\left(\frac{m\pi z}{d}\right) Y_{l,p}^{(n-1)}(\boldsymbol{q}),$$

where $m, l \in \mathbb{N}_0$, $1 \leq p \leq M_l^{(n-1)}$. Since $v_{l,p,0,2} = 0$ we drop the final subscript in the case of $m = 0$ and set $v_{l,p,0,1}(\boldsymbol{q}, z) = v_{l,p,0}(\boldsymbol{q}, z) = Y_{l,p}^{(n-1)}(\boldsymbol{q})$. The spherical harmonics are dense in the continuous functions on $\mathscr{S}_R^{n-1}$ and $\mathscr{S}^1_{\frac{d}{\pi}}$, so the functions $v_{l,p,m,1}$ and $v_{l,p,m,2}$ are dense in the continuous functions $\mathscr{T}^n_{R,d}$. Hence we have completely characterised the spectrum of $\tilde{A}_t$. $\square$

Lemma 6.1.2. *The spectrum of $\partial G_t(0) : h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right) \subset h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right) \to h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ consists of a sequence of isolated eigenvalues given by:*

$$\sigma(\partial G_t(0)) = \left\{ -\frac{\frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)\, m^2\pi^2}{d^2} - \frac{\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\,(l-1)(l+n-1)}{R^2} : m,l \in \mathbb{N}_0, l+m \geq 1 \right\}, \tag{6.3}$$

with eigenfunctions:

$$v_{l,p,m,1}(\boldsymbol{q},z) = \cos\left(\frac{m\pi z}{d}\right) Y^{(n-1)}_{l,p}(\boldsymbol{q}),\ v_{l,p,m,1}(\boldsymbol{q},z) = \sin\left(\frac{m\pi z}{d}\right) Y^{(n-1)}_{l,p}(\boldsymbol{q}),$$

for $1 \leq p \leq M^{(n-1)}_l$, and $v_{1,0,0} = 1$.

Furthermore if

$$R > \frac{d}{\pi}\sqrt{\frac{\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\,(n-1)}{\frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)}} \tag{6.4}$$

then all eigenvalues are non-positive and zero is an isolated eigenvalue of multiplicity $n+1$ with a basis of the eigenspace being the zeroth and first order spherical harmonics on $\mathscr{S}^{n-1}_R$ as functions on $\mathscr{T}^n_{R,d}$.

Proof. We start by noting that again the spectrum must consist solely of eigenvalues and that $Y^{(n-1)}_{0,1} = 1$ is an eigenfunction of $\partial G_t(0)$ with eigenvalue zero; we label this eigenfunction $v_{1,0,0}$. Now we note that the operator $\partial G_t(0)$ is self adjoint with respect to the L^2-inner product on $h^{0,\alpha}\left(\mathscr{T}^n_{R,d}\right)$. To see this, consider $v, w \in h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$ and compute:

$$\begin{aligned}
\int_{\mathscr{T}^n_{R,d}} \partial G_t(0)[v] w \, d\mu_0 &= \int_{\mathscr{T}^n_{R,d}} \left(\tilde{A}_t[v] - \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n-1}{R^2} \fint_{\mathscr{T}^n_{R,d}} v \, d\mu_0 \right) w \, d\mu_0 \\
&= \int_{\mathscr{T}^n_{R,d}} \tilde{A}_t[v] w \, d\mu_0 - \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n-1}{R^2} \fint_{\mathscr{T}^n_{R,d}} v \, d\mu_0 \int_{\mathscr{T}^n_{R,d}} w \, d\mu_0 \\
&= \int_{\mathscr{T}^n_{R,d}} v \tilde{A}_t[w] \, d\mu_0 - \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n-1}{R^2} \int_{\mathscr{T}^n_{R,d}} v \, d\mu_0 \fint_{\mathscr{T}^n_{R,d}} w \, d\mu_0,
\end{aligned}$$

where we use that $\tilde{A}_t$ is self adjoint with respect to the L^2-inner product. Hence:

$$\begin{aligned}
\int_{\mathscr{T}^n_{R,d}} \partial G_t(0)[v] w \, d\mu_0 &= \int_{\mathscr{T}^n_{R,d}} v \left(\tilde{A}_t[w] - \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n-1}{R^2} \fint_{\mathscr{T}^n_{R,d}} w \, d\mu_0 \right) d\mu_0 \\
&= \int_{\mathscr{T}^n_{R,d}} v \partial G_t(0)[w] \, d\mu_0.
\end{aligned}$$

Therefore we need only consider eigenfunctions that are L^2-orthogonal to $Y_{0,1}^{(n-1)}$ in order to characterise the remainder of the spectrum. This means that for an eigenfunction v with eigenvalue λ we assume the property:

$$\int_{\mathscr{T}^n_{R,d}} v \, d\mu_0 = 0,$$

and hence

$$\lambda v = \partial G_t(0)[v] = \tilde{A}_t[v].$$

Thus the remaining eigenfunctions of $\partial G_t(0)$ are precisely the remaining eigenfunctions of $\tilde{A}_t$, which are given in Lemma 6.1.1.

We now consider the sign of the eigenvalues as given in (6.3). It is clear that $\lambda_{l,m}$ is strictly decreasing in both l and m and we have that $\lambda_{1,0} = 0$ while

$$\lambda_{0,1} = -\left(\frac{\frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)\, \pi^2}{d^2} - \frac{\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\,(n-1)}{R^2} \right).$$

Therefore, under the assumption (6.4), the only non-negative eigenvalue is $\lambda_{1,0} = 0$ and it has multiplicity $1 + M_1^{(n-1)} = n+1$. $\square$

Note that if $R = \frac{d}{\pi}\sqrt{\frac{\frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)(n-1)}{\frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)}}$ then all the eigenvalues remain non-positive. However, we exclude this case for reasons that will be discussed in Section 7.2.

6.2 Center Manifold

Much of the remainder of this chapter follows the work set out in Chapter 5. We consider the local system:

$$\begin{aligned} x'(t) &= \partial G_t(0)_+[x(t)] + P_+\left[\bar{G}_t\left(\eta\left(\tfrac{x(t)}{r}\right) + y(t)\right)\right], & x(0) &= P_+[u_0], \\ y'(t) &= \partial G_t(0)_-[y(t)] + (I - P_+)\left[\bar{G}_t\left(\eta\left(\tfrac{x(t)}{r}\right) + y(t)\right)\right], & y(0) &= (I-P_+)[u_0], \end{aligned} \tag{6.5}$$

for $u_0 \in h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$.

Theorem 6.2.1. *Assuming the condition (6.4), there exists $\tilde{R}_2 > 0$ such that for any $r \in (0, \tilde{R}_2]$ there is a function $\gamma_r \in C^{1,1}\left(X^c, X^s_{2,\alpha}\right)$ such that $\gamma_r(0) = 0$ and $\partial\gamma_r(0) = 0$. Further, if $u_0 \in \mathcal{M}^c_r :=$ graph(γ_r) then the solution to (6.1), $u(t)$ is in $\mathcal{M}^c_r$ as long as $P[u(t)] \in B_{X^c,r}(0)$. The dimension of $\mathcal{M}^c_r$ is $n+1$.*

Again, since $\partial G_t(0)$ is self adjoint with respect to the L^2-inner product, it commutes with the L^2-orthogonal projection onto X^c:

$$P[u] := \sum_{a=0}^{n} \frac{\langle u, v_{1,a,0}\rangle}{\langle v_{1,a,0}, v_{1,a,0}\rangle} v_{1,a,0}. \tag{6.6}$$

That is, $P\left[\partial G_t(0)[u]\right] = \partial G_t(0)\left[P[u]\right] = 0$ for all $u \in h^{2,\alpha}\left(\mathscr{T}^n_{R,d}\right)$. Notably this means that $P\left[h^{k+2,\alpha}\left(\mathscr{T}^n_{R,d}\right)\right] = N\left(\partial G_t(0)\right)$ and $(I-P)\left[h^{k+2,\alpha}\left(\mathscr{T}^n_{R,d}\right)\right] = \operatorname{Range}\left(\partial G_t(0)\right)$, so $P = P_+$, the spectral projection associated to $\sigma_+(\partial G_t(0))$. This also means that $\partial G_t(0)_+ = 0$. Note that if we define $\bar{a}_{k,\alpha} := \sum_{a=0}^{n} \frac{\|v_{1,a,0}\|_{h^{k,\alpha}} \int_{\mathscr{T}^n_{R,d}} |v_{1,a,0}|\, d\mu_0}{\langle v_{1,a,0}, v_{1,a,0}\rangle}$ we obtain the same bounds as in (5.9):

$$\|P_+[u]\|_{h^{k,\alpha}} \leq \bar{a}_{k,\alpha}\, \|u\|_{C^0}\,, \;\; \|(I-P_+)[u]\|_{h^{k,\alpha}} \leq (1+\bar{a}_{k,\alpha})\|u\|_{h^{k,\alpha}}, \tag{6.7}$$

where $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$

We set

$$\mathcal{C} := \left\{u \in V_{2,\alpha} : u = u_\rho \text{ and } \Omega_\rho \text{ is a cylinder}\right\},$$

and note that if $u \in \mathcal{C}$ then it is an equilibrium of (6.1), see (1.10) for the definition of u_ρ.

Lemma 6.2.2. *Assuming the condition (6.4), there exists a neighbourhood of zero, $W_c \subset X^c$, such that $\mathcal{M}^c_r$ and $\mathcal{C}$ are identical inside $(W_c \cap B_{X^c,r}(0)) \times B_{X^s_{2,\alpha},r}(0)$, for any $r \in (0, \tilde{R}_2]$.*

Proof. Firstly, since any $u_0 \in \mathcal{C} \cap \left(B_{X^c,r}(0) \times B_{X^s_{2,\alpha},r}(0)\right)$ is a stationary solution to (6.1), we use Corollary 5.2.3 to conclude that $u_0 \in \mathcal{M}^c_r$. The rest of the proof follows in a similar manner to Lemma 5.2.5: If $u \in \mathcal{C}$, then there exists a $\rho \in h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, independent of z, that describes a cylinder and such that $u = u_\rho$. From the graph of ρ we obtain the parameters $\boldsymbol{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$, where $y_0 := R' - R$, R' is the radius of Ω_ρ, and $(y_1, \ldots, y_n, 0)$ is the point in $\mathbb{R}^{n+1}$ where Ω_ρ's axis of rotation meets the $z = 0$ hyperplane. Since

$$\boldsymbol{X}_\rho = R\left(Y^{(n-1)}_{1,1}, \ldots, Y^{(n-1)}_{1,n}, \frac{z}{R}\right) + \rho\left(Y^{(n-1)}_{1,1}, \ldots, Y^{(n-1)}_{1,n}, 0\right)$$

we have the relationship:

$$(R+y_0)^2 = R'^2 = \sum_{a=1}^{n}\left((R+\rho)Y^{(n-1)}_{1,a} - y_a\right)^2. \tag{6.8}$$

This is essentially the same equation as (5.10) so we obtain

$$R+\rho=\sum_{a=1}^{n} y_a Y_{1,a}^{(n-1)}+\sqrt{\left(\sum_{a=1}^{n} y_a Y_{1,a}^{(n-1)}\right)^2+(R+y_0)^2-\sum_{a=1}^{n} y_a^2},$$

and if we set

$$\chi(\boldsymbol{y}):=\sum_{a=1}^{n} y_a v_{1,a,0}-R v_{1,0,0}+\sqrt{\left(\sum_{a=1}^{n} y_a v_{1,a,0}\right)^2+\left((R+y_0)^2-\sum_{a=1}^{n} y_a^2\right) v_{1,0,0}}, \quad (6.9)$$

we have $u=u_\rho=\chi(\boldsymbol{y})$. We also have $\chi: U\subset\mathbb{R}^{n+1}\to h^{2,\alpha}\left(\mathscr{T}_{R,d}^n\right)$, where U is a neighbourhood of zero, and it is clear from the construction that for any $u\in\mathcal{C}$, with sufficiently small norm, there exists a $\boldsymbol{y}\in U$ such that $u=\chi(\boldsymbol{y})$. This map is also smooth on U and we use equation (5.12) to obtain:

$$\partial\chi(0)[\boldsymbol{x}]=\sum_{a=0}^{n} x_a v_{1,a,0},$$

where $\boldsymbol{x}\in\mathbb{R}^{n+1}$.

Considering the map $\bar{\chi}(\boldsymbol{y}): U\to X^c$ given by $\bar{\chi}(\boldsymbol{y}):=P_+[\chi(\boldsymbol{y})]$, the linearisation at zero is given by $\partial\bar{\chi}(0)[\boldsymbol{x}]=\sum_{a=0}^{n} x_a v_{1,a,0}$ and hence is the identity map with respect to the basis $v_{1,p,0}$, $0\le p\le n$, of X^c. Therefore there exists a neighbourhood of zero $V\subset X^c$ such that $\bar{\chi}$ is a diffeomorphism from V onto its image, $W_c\subset X^c$. Further, the function $\bar{\gamma}:=\chi\circ\bar{\chi}^{-1}-I: W_c\to X^s_{2,\alpha}$ parametrises $\mathcal{C}$ as a graph over X^c locally. Since from the first remark of the proof we have that $\mathcal{C}\cap\left(B_{X^c,r}(c)\times B_{X^s_{2,\alpha},r}(0)\right)\subset\mathcal{M}^c_r$, we conclude that $\mathcal{C}$ and $\mathcal{M}^c_r$ coincide inside $(W_c\cap B_{X^c,r}(0))\times B_{X^s_{2,\alpha},r}(0)$. Note that we also have $\bar{\gamma}|_{W_c\cap B_{X^c,r}(0)}=\gamma_r|_{W_c\cap B_{X^c,r}(0)}$. □

6.3 Convergence to a Cylinder

In this section we prove the main result of the chapter, that the cylinders with large enough radius are stable under the weighted volume preserving curvature flows.

Theorem 6.3.1. *Assuming the condition (6.4), there exists a neighbourhood of zero $O_c\subset h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, $0<\alpha<1$, such that if $\rho_0\in O_c$, then the flow (1.4) with initial hypersurface Ω_{ρ_0} exists for all time. Furthermore, the hypersurfaces converge exponentially fast to a cylinder as $t\to\infty$, with respect to the $h^{2,\alpha}\left(\overline{\mathscr{C}}^n_{R,d}\right)$ topology, $\alpha\in(0,1)$.*

Proof. The proof is very similar to the proof of Theorem 5.3.2. For completeness we include the essential steps here. Again we fix $r \in (0, \tilde{R}_2]$ and, since the center manifold is comprised locally of stationary solutions to the flow, we have $w_r(t; x, \gamma_r) = x$ for all $x \in W_c \cap B_{X^c,r}(0)$, where $w_r(t; x, \gamma_r)$ solves

$$w'(t) = P_+\left[\bar{G}_t\left(\eta\left(\frac{w(t)}{r}\right) w(t) + \gamma_r(w(t))\right)\right],\ w(0) = x.$$

Also, by (6.7), if u_0 is small enough we can apply Proposition 5.3.1 with

$$\omega_- = \min\left(\frac{\partial F}{\partial \kappa_n}(\boldsymbol{\kappa}_0)\frac{\pi^2}{d^2} - \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n-1}{R^2}, \frac{\partial F}{\partial \kappa_1}(\boldsymbol{\kappa}_0)\frac{n+1}{R^2}\right), \tag{6.10}$$

to obtain, for any $\omega \in (0, \omega_-)$, $\bar{x} \in X^c$ such that:

$$\begin{aligned}\|x_r(t) - w_r(t; \bar{x}, \gamma_r)\|_{h^{0,\alpha}} + &\|y_r(t) - \gamma_r(w_r(t; \bar{x}, \gamma_r))\|_{h^{2,\alpha}} \\ &\le Ce^{-\omega t}\|(I - P_+)[u_0] - \gamma_r\left(P_+[u_0]\right)\|_{h^{2,\alpha}},\end{aligned} \tag{6.11}$$

where $(x_r(t), y_r(t))$ solves (6.5). By evaluating this at $t = 0$ we obtain the bound $\|\bar{x}\|_{h^{0,\alpha}} \le C\|u_0\|_{h^{2,\alpha}}$ as in (5.15). So if u_0 is small enough we have $\bar{x} \in W_c \cap B_{X^c,r}(0)$. Hence, by the first part of the proof $w_r(t; \bar{x}, \gamma_r) = \bar{x}$ and (6.11) simplifies to:

$$\|x_r(t) - \bar{x}\|_{h^{0,\alpha}} + \|y_r(t) - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \le Ce^{-\omega t}\|(I - P_+)[u_0] - \gamma_r\left(P_+[u_0]\right)\|_{h^{2,\alpha}}. \tag{6.12}$$

The bound $\|x_r(t)\|_{h^{0,\alpha}} \le C\|u_0\|_{h^{2,\alpha}}$ is then obtained by the same calculations as in (5.17) and, by considering u_0 small enough, we have $u(t) = x_r(t) + y_r(t)$ is a solution to (6.1) for all $t \ge 0$. Hence:

$$\begin{aligned}\|u(t) - (\bar{x} + \gamma_r(\bar{x}))\|_{h^{2,\alpha}} &= \|P_+[u(t)] - \bar{x} + (I - P_+)[u(t)] - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \\ &\le \|P_+[u(t)] - \bar{x}\|_{h^{2,\alpha}} + \|(I - P_+)[u(t)] - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \\ &\le C\|P_+[u(t)] - \bar{x}\|_{h^{0,\alpha}} + \|(I - P_+)[u(t)] - \gamma_r(\bar{x})\|_{h^{2,\alpha}} \\ &\le Ce^{-\omega t}\|(I - P_+)[u_0] - \gamma_r(P_+[u_0])\|_{h^{2,\alpha}},\end{aligned} \tag{6.13}$$

where we used equivalence of norms on X^c, and we obtain that $u(t)$ converges exponentially to $u_\infty := \bar{x} + \gamma_r(\bar{x}) \in \mathcal{M}_r^c$.

Finally, since $\|u_{\rho_0}\|_{h^{2,\alpha}}$ is controlled by $\|\rho_0\|_{h^{2,\alpha}}$ for any $\rho_0 \in h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, see Corollary 3.1.4, there exists a neighbourhood of zero such that if ρ_0 is in this neighbourhood then u_{ρ_0} is small and the above analysis is applicable. Therefore $\rho(t) = u(t)|_{\overline{\mathscr{C}}^n_{R,d}}$ converges exponentially fast to $u_\infty|_{\overline{\mathscr{C}}^n_{R,d}}$, which by Lemma 6.2.2 is a cylinder. $\square$

A direct consequence of the theorem is the existence of non-axially symmetric hypersurfaces that converge to a cylinder under the flow. We also have the following corollary concerning the stability of hypersurfaces that converge to cylinders under the flow.

Corollary 6.3.2. *Let $\rho(t)$ be a solution to the equation (1.5), with R satisfying (6.4), which exists for all time and converges to zero. Suppose further that $\frac{\partial F}{\partial \kappa_i}\left(\boldsymbol{\kappa}_{\rho(t)}\right) > 0$ for all $t \in [0, \infty)$ and $i = 1, \ldots, n$. Then there exists a neighbourhood, $O_{c,1} \subset h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, $0 < \alpha < 1$, of ρ_0 such that for every $v_0 \in O_{c,1}$ the solution to (1.5) with initial condition v_0 exists for all time and converges to a function near zero whose graph is a cylinder.*

Proof. This follows by analysing (1.7) using the same arguments given in Corollary 5.3.3 with the obvious changes. □

6. STABILITY OF WEIGHTED VOLUME PRESERVING CURVATURE FLOWS NEAR FINITE CYLINDERS

7

Stability of Volume Preserving Mean Curvature Flow near Finite Cylinders

Here we consider hypersurfaces that are close to a cylinder and evolve them using the volume preserving mean curvature flow. This is a special case of the problem considered in Chapter 6 and as such the results of that chapter are still applicable. In particular, we have shown there exists an exponentially attractive center manifold and if the initial hypersurface is $h^{2,\alpha}$-close to a cylinder, for any $\alpha \in (0,1)$, then it converges to a cylinder with respect to the $h^{2,\alpha}$ norm, under the assumption

$$R > \frac{d\sqrt{n-1}}{\pi}. \tag{7.1}$$

This assumption should be compared to the condition (1.3), which was used in [11] to prove convergence to cylinders in the case of axial symmetry. In the case of the hypersurface being a cylinder the assumption (1.3) reduces to $R \geq nd$. Since the right hand side is strictly greater than $\frac{d\sqrt{n-1}}{\pi}$, Theorem 6.3.1 shows that (1.3) can be relaxed by assuming the axially symmetric hypersurfaces are close to a cylinder. The condition (7.1) also appears in [9], which proves that two dimensional cylinders of large radius are stable solutions to the isoperimetric problem.

In this chapter we extend this result to include initial hypersurfaces that are $h^{1,\beta}$-close to a cylinder, for any $\beta \in (0,1)$. The existence of solutions to the flow with such an initial condition was proved in Theorem 4.4.2. We also have that the flow becomes smooth instantaneously (Corollary 4.4.3) and we will find that this allows us to obtain convergence to a cylinder in the C^k-topology for any $k \in \mathbb{N}$. The last two sections of this chapter deal with condition (7.1). It will be investigated through a bifurcation analysis of the stationary solution equation and from a geometric point of view.

7.1 Smooth Convergence to a Cylinder

In this section we prove the convergence of solutions of the volume preserving mean curvature flow to cylinders if the initial height function is small in $h^{1,\beta}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, for any $\beta \in (0,1)$. The results of this section are also included in [28]. We follow [20] in using results presented in [42]. If, in addition to the assumptions in Theorem 4.4.1, we assume that $\sigma_+(A) \subset i\mathbb{R}$, then we have the following results:

Proposition 7.1.1 (Proposition 5.4 [42]). *Let $\omega_c \in (0, \omega_-)$ and consider equations (5.3) and (5.5) with $A = -Q(0) - \partial f(0)$ and $G(u) = -Q(u)[u] + f(u) - A[u]$, see also (4.14). There exists $R' > 0$ such that for every $r \in (0, R']$ there exists a $K_r \in \mathbb{R}^+$, with $\lim_{r\to 0} K_r = 0$, and $\tilde{W}_{k+1,\beta} \subset h^{k+1,\beta}\left(\mathscr{T}^n_{R,d}\right)$ a neighbourhood of zero, such that if $u_0 \in \tilde{W}_{k+1,\beta}$ then the solution to (5.3) with $x_0 = P_+[u_0]$ and $y_0 = (I - P_+)[u_0]$ satisfies*

$$\|x_r(\tau) - \tilde{w}_r(\tau,t)\|_{h^{0,\alpha}} \le K_r \int_\tau^t e^{(K_r+\omega_c)(s-\tau)} \|y_r(s) - \gamma_r(x(s))\|_{h^{k+2,\alpha}}\, ds,\ 0 < \tau \le t < \delta,$$

where $\tilde{w}_r(\tau,t) := w_r(\tau - t; x_r(t), \gamma_r)$ for $\tau \in \mathbb{R}$, $t \in [0,\delta)$. Furthermore

$$\begin{aligned}&\|w_r(\tau; P_+[u_0], \gamma_r) - \tilde{w}_r(\tau,t)\|_{h^{0,\alpha}}\\ &\qquad \le K_r e^{-(K_r+\omega_c)\tau} \int_0^t e^{(K_r+\omega_c)s} \|y_r(s) - \gamma_r(x_r(s))\|_{h^{2,\alpha}}\, ds,\ \tau \le 0 \le t < \delta.\end{aligned}$$

Theorem 7.1.2 (Theorem 5.8 [42]). *There exists $\bar{R} \in (0, R']$ such that for all $r \in (0, \bar{R}]$, $K_r + \omega_c < \omega_-$ and the solution to (5.3) with $x_0 = P_+[u_0]$, $y_0 = (I - P_+)[u_0]$ satisfies*

$$\|y_r(t) - \gamma_r(x_r(t))\|_{h^{k+2,\alpha}} \le \frac{Ce^{-\omega t}}{t^{\frac{1-\beta+\alpha}{2}}} \|(I-P_+)[u_0] - \gamma_r\left(P_+[u_0]\right)\|_{h^{k+1,\beta}},\ t \in (0, t^+(u_0))$$

for any $\omega \in (K_r + \omega_c, \omega_-)$ and each initial value $u_0 \in \tilde{W}_{k+1,\beta}$. Note that C only depends on the difference $\beta - \alpha$.

We now fix $l \in \mathbb{N}_0$, $\bar{\alpha} \in (0,1)$, $\beta_0 \in (\bar{\alpha}, 1)$, $\omega_c \in (0, \omega_-)$, see (6.10), and define $\beta_k = \beta_0 - \frac{k(\beta_0 - \bar{\alpha})}{l+1}$. Let $\bar{R}_l$ be the constant from applying Theorem 7.1.2 to the system (6.5) with $k = l$, $\beta = \beta_l$ and $\alpha = \bar{\alpha}$, and fix $r \in (0, \bar{R}_l]$, $\omega \in (K_r + \omega_c, \omega_-)$. The aim for the remainder of this section is to find a set $W_l \subset h^{1,\beta_0}\left(\mathscr{T}^n_{R,d}\right)$ such that if $u_0 \in W_l$ then the solution, $u(t)$, to (1.9) exists for all time and converges to an element in $\mathcal{M}^c_r$, defined in Theorem 6.2.1.

To achieve this, we first note that, assuming (7.1), we can apply Theorem 5.2.1 and Corollary 5.2.3 to obtain functions $\gamma_{k,r} \in C^{1,1}\left(X^c, X^s_{k+2,\beta_{k+1}}\right)$ for all $0 \le k \le l$, we set b_r to be the maximum of their Lipschitz constants. The arguments in Lemma 6.2.2 are still valid for each $\gamma_{k,r}$ and hence $\bar{\gamma}|_{W_c \cap B_{X^c,r}(0)} = \gamma_{k,r}|_{W_c \cap B_{X^c,r}(0)}$. Therefore, when we work on $W_c \cap B_{X^c,r}(0)$ we denote all maps by $\bar{\gamma}$.

We now obtain bounds for how much $x_r(t)$ can grow over a short time period.

Lemma 7.1.3. *There exists a neighbourhood of zero, $U \subset h^{1,\beta_0}\left(\mathscr{T}^n_{R,d}\right)$, $\bar{\tau} > 0$ and $C > 0$ such that if $u_0 \in U$ then*

$$\|x_r(t) - P_+[u_0]\|_{h^{0,\beta_1}} \le C\|(I - P_+)[u_0] - \bar{\gamma}(P_+[u_0])\|_{h^{1,\beta_0}},\ t \in [0, \bar{\tau}].$$

where C depends on the choices of $\bar{\alpha}$, β_0, l, r, ω_c and ω.

Proof. The mapping $(t, u_0) \to u(t)$ is a continuous semiflow by Theorem 4.4.2. Hence, we can find $\bar{\tau} \in (0, \delta)$ and a neighbourhood of zero $U \subset \tilde{W}_{1,\beta_0}$ such that if $u_0 \in U$ then $u(t) \in (W_c \cap B_{X^c,r}(0)) \times B_{X^s_{1,\beta_0},r}(0)$ for all $t \in [0, \bar{\tau}]$. In particular, this means that $P_+[u(t)] \in W_c \cap B_{X^c,r}(0)$ for all $t \in [0, \bar{\tau}]$ so that (1.9) and (6.5) are equivalent on this time interval. Therefore, $u(t) = x_r(t) + y_r(t)$, $w_r(\tau; x_r(t), \bar{\gamma}) = x_r(t)$ and $\tilde{w}_r(\tau, t) = x_r(t)$ for all $t \in [0, \bar{\tau}]$ and $\tau \in \mathbb{R}$.

We now set $\tau = 0$ in the second estimate in Proposition 7.1.1 then apply Theorem 7.1.2 to obtain for $t \in [0, \bar{\tau}]$:

$$\begin{aligned}
\|P_+[u_0] - x_r(t)\|_{h^{0,\beta_1}} &\le K_r \int_0^t e^{(K_r+\omega_c)s}\|y_r(s) - \bar{\gamma}(x_r(s))\|_{h^{2,\beta_1}}\, ds,\\
&\le K_r C \int_0^t \frac{e^{-(\omega-(K_r+\omega_c))s}}{s^{\frac{1-\beta_0+\beta_1}{2}}}\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\|_{h^{1,\beta_0}}\, ds\\
&\le \frac{K_r C \Gamma\left(\frac{1+\beta_0-\beta_1}{2}\right)}{(\omega - (K_r+\omega_c))^{\frac{1+\beta_0-\beta_1}{2}}}\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\|_{h^{1,\beta_0}},
\end{aligned}$$

where $\Gamma(x)$ is the gamma function. □

This allows us to obtain convergence in the h^{2,β_1} norm.

Lemma 7.1.4. *There exists a neighbourhood of zero, $V \subset h^{1,\beta_0}\left(\mathscr{T}^n_{R,d}\right)$, and $\bar{\tau} > 0$ such that if $u_0 \in V$ then the flow (1.9) has a solution for all $t \ge 0$ and the solution $u(t)$ satisfies*

$$\|u(t) - (\bar{x} + \bar{\gamma}(\bar{x}))\|_{h^{2,\beta_1}} \le Ce^{-\omega t}\|(I - P_+)[u_0] - \bar{\gamma}(P_+[u_0])\|_{h^{1,\beta_0}},\ t \ge \bar{\tau},$$

for some $\bar{x} \in W_c \cap B_{X^c,r}(0)$. Here C depends on r, $\bar{\alpha}$, β_0, l, and ω.

Proof. We consider U and $\bar{\tau}$ as given in Lemma 7.1.3 and proceed to bound $u(\bar{\tau})$, when $u_0 \in U$. Using $u(\bar{\tau}) = x_r(\bar{\tau}) + y_r(\bar{\tau})$:

$$\begin{aligned}\|u(\bar{\tau})\|_{h^{2,\beta_1}} &\leq \|x_r(\bar{\tau})\|_{h^{2,\beta_1}} + \|y_r(\bar{\tau}) - \bar{\gamma}(x_r(\bar{\tau}))\|_{h^{2,\beta_1}} + \|\bar{\gamma}(x_r(\bar{\tau}))\|_{h^{2,\beta_1}} \\ &\leq (C + b_r)\|x_r(\bar{\tau})\|_{h^{0,\beta_1}} + \frac{Ce^{-\omega\bar{\tau}}}{\bar{\tau}^{\frac{1-\beta_0+\beta_1}{2}}}\|(I - P_+)[u_0] - \bar{\gamma}(P_+[u_0])\|_{h^{1,\beta_0}},\end{aligned}$$

where we have used that $\bar{\gamma}$ is Lipschitz, the equivalence of norms on X^c and Theorem 7.1.2. Continuing, via Lemma 7.1.3, we have:

$$\begin{aligned}\|u(\bar{\tau})\|_{h^{2,\beta_1}} \leq& C\left(\|x_r(\bar{\tau}) - P_+[u_0]\|_{h^{0,\beta_1}} + \|P_+[u_0]\|_{h^{0,\beta_1}}\right) \\ &+ \frac{Ce^{-\omega\bar{\tau}}}{\bar{\tau}^{\frac{1-\beta_0+\beta_1}{2}}}\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\|_{h^{1,\beta_0}} \\ \leq& \left(C + \frac{Ce^{-\omega\bar{\tau}}}{\bar{\tau}^{\frac{1-\beta_0+\beta_1}{2}}}\right)\|(I - P_+)[u_0] - \bar{\gamma}(P_+[u_0])\|_{h^{1,\beta_0}} + C\|P_+[u_0]\|_{h^{0,\beta_1}} \\ \leq& C(\bar{\tau}, \bar{\alpha}, \beta_0, l, \omega)\|u_0\|_{h^{1,\beta_0}}. \end{aligned} \tag{7.2}$$

Therefore there exists $V \subset U$ such that if $u_0 \in V$, then $u(\bar{\tau})$ is close to zero in $h^{2,\beta_1}\left(\mathscr{T}^n_{R,d}\right)$. Hence we can apply the result in the proof of Theorem 6.3.1 to conclude that the solution, $\bar{u}(t)$, to the flow (1.9) together with the initial condition $u(\bar{\tau})$ exists, $P_+[\bar{u}(t)] \in W_c \cap B_{X^c,r}(0)$ for all time, and $\bar{u}(t)$ satisfies equation (6.13), i.e.

$$\|\bar{u}(t) - (\bar{x} + \bar{\gamma}(\bar{x}))\|_{h^{2,\beta_1}} \leq Ce^{-\omega t}\|(I - P_+)[u(\bar{\tau})] - \bar{\gamma}\left(P_+[u(\bar{\tau})]\right)\|_{h^{2,\beta_1}},\ t \geq 0$$

for some $\bar{x} \in W_c \cap B_{X^c,r}(0)$. However, by uniqueness of the flow, we also have that $\bar{u}(t) = u(t + \bar{\tau})$ for $t \geq 0$, so, using the transformation $t \mapsto t - \bar{\tau}$, we obtain for $t \geq \bar{\tau}$:

$$\begin{aligned}\|u(t) - (\bar{x} + \bar{\gamma}(\bar{x}))\|_{h^{2,\beta_1}} &\leq Ce^{-\omega(t-\bar{\tau})}\|(I - P_+)[u(\bar{\tau})] - \bar{\gamma}\left(P_+[u(\bar{\tau})]\right)\|_{h^{2,\beta_1}} \\ &\leq \frac{C}{\bar{\tau}^{\frac{1-\beta_0+\beta_1}{2}}}e^{-\omega t}\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\|_{h^{1,\beta_0}},\end{aligned}$$

where we again used Theorem 7.1.2. □

Note this theorem provides stability of cylinders under perturbations in h^{1,β_0}, but before stating this result we obtain higher convergence for the solution. Furthermore, we now have, due to comments in the proofs of Lemma 7.1.3 and Theorem 7.1.4, that if $u_0 \in V$ then $P_+[u(t)] \in W_c \cap B_{X^c,r}(0)$ for all $t \geq 0$. Hence, $u(t) = x_r(t) + y_r(t)$, $w_r(\tau; x_r(t), \bar{\gamma}) = x_r(t)$ and $\tilde{w}_{l,r}(\tau, t) = x_r(t)$ for all $t \geq 0$ and $\tau \in \mathbb{R}$.

Since we have convergence of the solution, we can obtain a bound independent of the time $\bar{\tau}$, we follow [20] to achieve this.

Lemma 7.1.5. *If $u_0 \in V$ then for all $t > 0$ we have*

$$\|u(t) - (\bar{x} + \bar{\gamma}(\bar{x}))\|_{h^{2,\beta_1}} \leq \frac{Ce^{-\omega t}}{t^{\frac{1-\beta_0+\beta_1}{2}}} \|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\|_{h^{1,\beta_0}},$$

where C depends on the choices of $\bar{\alpha}$, β_0, l, r, ω_c and ω.

Proof. Using that $\tilde{w}_{l,r}(\tau, t) = x_r(t)$, the first bound in Proposition 7.1.1 simplifies to:

$$\|x_r(\tau) - x_r(t)\|_{h^{0,\beta_1}} \leq K_r \int_\tau^t e^{(K_r+\omega_c)(s-\tau)} \|y_r(s) - \bar{\gamma}(x_r(s))\|_{h^{2,\beta_1}}\, ds,$$

for $0 < \tau \leq t$. We use Lemma 7.1.4 together with the bound for $\bar{x}$ in the proof of Theorem 6.3.1 and equation (7.2) to obtain the bound $\|u(t)\|_{h^{2,\beta_1}} \leq C\|u_0\|_{h^{1,\beta_0}}$ for $t \geq \bar{\tau}$. This, together with the flow being continuous on h^{1,β_0}, means we can ensure $u(\tau) \in \tilde{W}_{1,\beta_0}$ for all $\tau \geq 0$, shrinking V if necessary. We can therefore apply Theorem 7.1.2 to the function $\tilde{u}(s) = u(s + \tau)$:

$$\begin{aligned}
\|x_r(\tau) - x_r(t)\|_{h^{0,\beta_1}} &\leq K_r \int_\tau^t e^{(K_r+\omega_c)(s-\tau)} \|\tilde{y}_r(s-\tau) - \bar{\gamma}(\tilde{x}_r(s-\tau))\|_{h^{2,\beta_1}}\, ds \\
&\leq CK_r \int_\tau^t \frac{e^{(K_r+\omega_c)(s-\tau)}}{(s-\tau)^{\frac{1-\beta_0+\beta_1}{2}}} e^{-\omega(s-\tau)} \|\tilde{y}_r(0) - \bar{\gamma}(\tilde{x}_r(0))\|_{h^{1,\beta_0}}\, ds \\
&= CK_r \|y_r(\tau) - \bar{\gamma}(x_r(\tau))\|_{h^{1,\beta_0}} \int_\tau^t \frac{e^{(K_r+\omega_c-\omega)(s-\tau)}}{(s-\tau)^{\frac{1-\beta_0+\beta_1}{2}}}\, ds \\
&\leq \frac{CK_r \Gamma\left(\frac{1+\beta_0-\beta_1}{2}\right)}{(\omega - (K_r+\omega_c))^{\frac{1-\beta_0+\beta_1}{2}}} \|y_r(\tau) - \bar{\gamma}(x_r(\tau))\|_{h^{1,\beta_0}}.
\end{aligned}$$

As the right hand side is independent of t we can take it to infinity and, using that Theorem 7.1.4 implies that $\lim_{t\to\infty} x_r(t) = \bar{x}$, obtain a bound for $\tau > 0$:

$$\|x_r(\tau) - \bar{x}\|_{h^{0,\beta_1}} \leq C\|y_r(\tau) - \bar{\gamma}(x_r(\tau))\|_{h^{1,\beta_0}}. \tag{7.3}$$

Note that this has a very similar form to the bound in Lemma 7.1.3, except this is valid for all $\tau > 0$ and bounds the distance to the limiting function, instead of the initial function. Finally we achieve the bound:

$$\begin{aligned}
\|u(t) - (\bar{x} + \gamma(\bar{x}))\|_{h^{2,\beta_1}} \leq& \|x_r(t) - \bar{x}\|_{h^{2,\beta_1}} + \|y_r(t) - \bar{\gamma}(\bar{x})\|_{h^{2,\beta_1}} \\
\leq& C\|x_r(t) - \bar{x}\|_{h^{0,\beta_1}} + \|y_r(t) - \bar{\gamma}(x_r(t))\|_{h^{2,\beta_1}} \\
&+ \|\bar{\gamma}(x_r(t)) - \bar{\gamma}(\bar{x})\|_{h^{2,\beta_1}} \\
\leq& (C + b_r)\|x_r(t) - \bar{x}\|_{h^{0,\beta_1}} + \|y_r(t) - \bar{\gamma}(x_r(t))\|_{h^{2,\beta_1}} \\
\leq& C\|y_r(t) - \bar{\gamma}(x_r(t))\|_{h^{2,\beta_1}},
\end{aligned} \tag{7.4}$$

and hence using Theorem 7.1.2 we obtain the result. ∎

To obtain convergence with respect to norms of higher regularity we first need another lemma.

Lemma 7.1.6. *For any $k \in \mathbb{N}_0 \cap [0, l]$ there exists a U_k such that if $u_0 \in U_k$ then for $t \geq t_k := \frac{(k+1)\bar{\tau}}{l+1}$*

$$\left\|(I - P_+)\left[u(t)\right] - \bar{\gamma}\left(P_+\left[u(t)\right]\right)\right\|_{h^{k+2,\beta_{k+1}}} \leq C_k e^{-\omega t}\left\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}}, \tag{7.5}$$

where $\bar{\tau} > 0$ is given in Lemma 7.1.4, and $C_k = C^{k+1}\left(\frac{l+1}{\bar{\tau}}\right)^{\frac{k+1}{2}\left(1-\frac{\beta_0-\bar{\alpha}}{l+1}\right)}$, C is constant in Theorem 7.1.2.

Proof. The base case is easily achieved, with $U_0 = U$, by using Theorem 7.1.2:

$$\begin{aligned}
\left\|(I - P_+)\left[u(t)\right] - \bar{\gamma}\left(P_+\left[u(t)\right]\right)\right\|_{h^{2,\beta_1}} \leq & \frac{Ce^{-\omega t}}{t^{\frac{1-\beta_0+\beta_1}{2}}}\left\|(I - P_+)\left[u_0\right] - \bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}} \\
\leq & \frac{Ce^{-\omega t}}{t_0^{\frac{1}{2}\left(1-\frac{\beta_0-\bar{\alpha}}{l+1}\right)}}\left\|(I - P_+)\left[u_0\right] - \bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}},
\end{aligned}$$

for $t \geq t_0$.

Now we assume that (7.5) is true for some $k \leq l - 1$ and we prove it is true for $k + 1$. Firstly we bound the solution at the time t_k using (7.5):

$$\begin{aligned}
\left\|u(t_k)\right\|_{h^{k+2,\beta_{k+1}}} \leq & \left\|x_r(t_k)\right\|_{h^{k+2,\beta_{k+1}}} + \left\|\bar{\gamma}\left(x_r(t_k)\right)\right\|_{h^{k+2,\beta_{k+1}}} \\
& + \left\|y_r(t_k) - \bar{\gamma}\left(x_r(t_k)\right)\right\|_{h^{k+2,\beta_{k+1}}} \\
\leq & C_k e^{-\omega t_k}\left\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}} + (C + b_r)\left\|x_r(t_k)\right\|_{h^{0,\beta_1}} \\
\leq & C_k e^{-\omega t_k}\left\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}} \\
& + C\left(\left\|x_r(t_k) - P_+[u_0]\right\|_{h^{0,\beta_1}} + \left\|P_+[u_0]\right\|_{h^{0,\beta_1}}\right) \\
\leq & \left(C_k e^{-\omega t_k} + C\right)\left\|(I - P_+)[u_0] - \bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}} \\
& + \bar{a}_{0,\beta_1}\left\|u_0\right\|_{h^{1,\beta_0}} \\
\leq & \left(\left(C_k e^{-\omega t_k} + C\right)\left(1 + \bar{a}_{1,\beta_0} + b_r \bar{a}_{0,\beta_1}\right) + \bar{a}_{0,\beta_1}\right)\left\|u_0\right\|_{h^{1,\beta_0}},
\end{aligned} \tag{7.6}$$

where to obtain the second last bound we used Lemma 7.1.3. Therefore we can make U_{k+1} small enough such that $u(t_k) \in \tilde{W}_{k+2,\beta_{k+1}}$ and hence we obtain a solution to (1.9), $\bar{u}(t) \in h^{k+3,\beta_{k+2}}\left(\mathscr{T}^n_{R,d}\right)$, such that $\bar{u}(0) = u(t_k)$ and it satisfies the bound in Theorem 7.1.2. Now by uniqueness we have that $\bar{u}(t) = u(t + t_k)$ for $t \geq 0$. So for $t \geq t_{k+1}$ we have

$$\begin{aligned}
&\left\|(I-P_+)\left[u(t)\right]-\bar{\gamma}\left(P_+\left[u(t)\right]\right)\right\|_{h^{k+3,\beta_{k+2}}} \\
&\qquad \leq \frac{Ce^{-\omega(t-t_k)}}{(t-t_k)^{\frac{1-\beta_{k+1}+\beta_{k+2}}{2}}}\left\|(I-P_+)\left[u(t_k)\right]-\bar{\gamma}\left(P_+\left[u(t_k)\right]\right)\right\|_{h^{k+2,\beta_{k+1}}} \\
&\qquad \leq \frac{CC_ke^{-\omega t}}{t_0^{\frac{1}{2}\left(1-\frac{\beta_0-\bar{\alpha}}{l+1}\right)}}\left\|(I-P_+)\left[u_0\right]-\bar{\gamma}\left(P_+\left[u_0\right]\right)\right\|_{h^{1,\beta_0}}, \qquad (7.7)
\end{aligned}$$

where we have once again used the inductive assumption (7.5). This then proves the bound (7.5) for all $k \in \mathbb{N}_0 \cap [0, l]$. □

We are now able to obtain convergence to $\bar{x} + \bar{\gamma}(\bar{x})$ in $h^{l+2,\bar{\alpha}}$.

Theorem 7.1.7. *For any $l \in \mathbb{N}_0$, $\bar{\alpha} \in (0,1)$, $\beta_0 \in (\bar{\alpha}, 1)$ there exists a neighbourhood of zero, $W_l \subset h^{1,\beta_0}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}}^n_{R,d}\right)$, such that if $\rho_0 \in W_l$ then its flow by (1.6) exists for all time and converges exponentially fast in C^{l+2} to the height function ρ_∞, where Ω_{ρ_∞} is a cylinder.*

Proof. By Corollary 3.1.4 we can choose W_l such that if $\rho_0 \in W_l$ then $u_{\rho_0} \in U_l$ and thus have the result in Lemma 7.1.6. In particular for $t > \bar{\tau}$ we have, using (7.3):

$$\begin{aligned}
\left\|u(t)-(\bar{x}+\bar{\gamma}(\bar{x}))\right\|_{h^{l+2,\bar{\alpha}}} \leq& \left\|x_r(t)-\bar{x}\right\|_{h^{l+2,\bar{\alpha}}}+\left\|y_r(t)-\bar{\gamma}\left(x_r(t)\right)\right\|_{h^{l+2,\bar{\alpha}}} \\
&+\left\|\bar{\gamma}\left(x_r(t)\right)-\bar{\gamma}(\bar{x})\right\|_{h^{l+2,\bar{\alpha}}} \\
\leq&(C+b_r)\left\|x_r(t)-\bar{x}\right\|_{h^{0,\beta_1}}+\left\|y_r(t)-\bar{\gamma}\left(x_r(t)\right)\right\|_{h^{l+2,\bar{\alpha}}} \\
\leq&C\left\|y_r(t)-\bar{\gamma}\left(x_r(t)\right)\right\|_{h^{l+2,\bar{\alpha}}} \\
\leq&CC_le^{-\omega t}\left\|(I-P_+)[u_0]-\bar{\gamma}\left(P_+[u_0]\right)\right\|_{h^{1,\beta_0}}. \qquad (7.8)
\end{aligned}$$

Therefore, we have shown that $\rho(t) = u(t)|_{\overline{\mathscr{C}}^n_{R,d}}$ converges exponentially fast, in C^l, to $\rho_\infty := (\bar{x} + \bar{\gamma}(\bar{x}))\,|_{\overline{\mathscr{C}}^n_{R,d}}$, which by Lemma 6.2.2 is the height function for a cylinder. □

7.2 Bifurcation Analysis

In Section 6.1 it was found that for the eigenvalues of the linearisation of $G_t(u)$ about zero to be non-positive, the radius of the cylinder needs to satisfy the condition

$$R \geq \frac{d}{\pi}\sqrt{\frac{\frac{\partial F}{\partial \kappa_1}\left(\boldsymbol{\kappa}_0\right)(n-1)}{\frac{\partial F}{\partial \kappa_n}\left(\boldsymbol{\kappa}_0\right)}}.$$

However we excluded the case of equality and instead assumed the radius satisfied the strict inequality. In this section we discover that any neighbourhood of the cylinder with radius $R = \frac{d\sqrt{n-1}}{\pi}$ contains constant mean curvature (CMC) unduloids, which do not converge to a cylinder under the volume preserving mean curvature flow as they are stationary solutions. Therefore the strict inequality was indeed necessary to obtain Theorems 6.3.1 and 7.1.7. The axially symmetric volume preserving mean curvature flow is equivalent to the PDE:

$$\frac{\partial u}{\partial t} = G(u) := \sqrt{1+\left(\frac{\partial u}{\partial z}\right)^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} H(u)\,d\mu_u - H(u)\right), \tag{7.9}$$

$$H(u) := \frac{-\frac{d^2u}{dz^2}}{\left(1+\left(\frac{du}{dz}\right)^2\right)^{3/2}} + \frac{n-1}{u\sqrt{1+\left(\frac{du}{dz}\right)^2}}, \tag{7.10}$$

Note that $H(u)$ is the mean curvature of the hypersurface obtained by rotating the graph of the function $u(z)$ around the z-axis and $d\mu_u = \mu(u)\,dz = \sqrt{1+\left(\frac{du}{dz}\right)^2}u^{n-1}\,dz$. We have removed the presence of R from the equation since we will be considering the flow near cylinders of various radii. We seek solutions in the space

$$h^{2,\alpha}_e\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) := \left\{u \in h^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) : u(z) = u(-z)\right\}.$$

As in [35], we use that the flow preserves enclosed volume to obtain an equivalent PDE on the space of average zero functions:

$$h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) := \left\{v \in h^{2,\alpha}_e\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) : \int_{\mathscr{S}^1_{\frac{d}{\pi}}} v(z)\,dz = 0\right\}.$$

To simplify notation we will define the projection operator:

$$P_0 : h^{2,\alpha}_e\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) \to h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right),\ P_0[u] := u - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} u\,dz. \tag{7.11}$$

Before we are able to state the equivalent flow, we require a function that recreates a function $u \in h^{2,\alpha}_e\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$ given its projection $P_0[u]$ and the enclosed volume of its corresponding hypersurface, $Vol(u)$.

Lemma 7.2.1. *For each $\eta_0 \in \mathbb{R}^+$ there exist V_{η_0}, a neighbourhood of the constant function $\frac{n-1}{\eta_0} \in h^{2,\alpha}_e\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$, and U_{η_0}, a neighbourhood of $(0,\eta_0) \in h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) \times \mathbb{R}$, as well as a smooth diffeomorphism $\psi_{\eta_0} : U_{\eta_0} \to V_{\eta_0}$, see Figure 7.1, such that for all $(\bar{u},\eta) \in U_{\eta_0}$ we have $P_0\left[\psi_{\eta_0}(\bar{u},\eta)\right] = \bar{u}$ and $Vol\left(\psi_{\eta_0}(\bar{u},\eta)\right) = 2\omega_n d\left(\frac{n-1}{\eta}\right)^n$, where ω_n is the volume of the unit n-ball.*

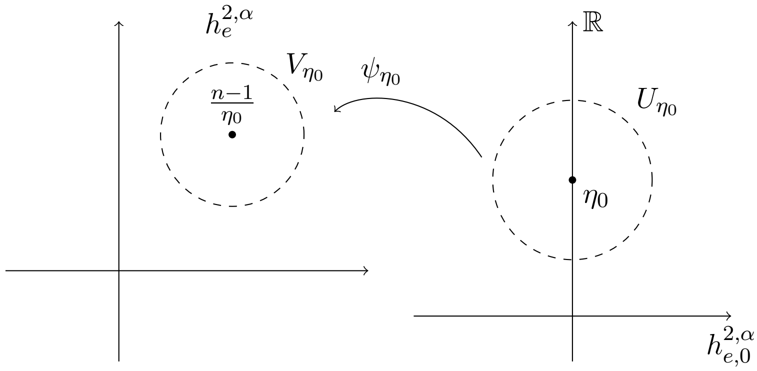


Figure 7.1: Mapping between zero mean functions and graph functions

Proof. We consider the function

$$\Phi(u,\bar{u},\eta) := \left(P_0[u]-\bar{u}, \omega_n d\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} u^n\,dz - \left(\frac{n-1}{\eta}\right)^n\right)\right). \tag{7.12}$$

We note that the points $\left(\frac{n-1}{\eta},0,\eta\right)$ are zeros of Φ and we calculate its linearisation with respect to the first argument:

$$\partial_1\Phi(u,\bar{u},\eta)[v] = \left(P_0[v], n\omega_n d\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} u^{n-1}v\,dz\right).$$

Evaluating at the point $\left(\frac{n-1}{\eta_0},0,\eta_0\right)$, for any $\eta_0\in\mathbb{R}^+$, gives:

$$\partial_1\Phi\left(\frac{n-1}{\eta_0},0,\eta_0\right)[v] = \left(P_0[v], n\omega_n d\left(\frac{n-1}{\eta_0}\right)^{n-1}\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} v\,dz\right).$$

If v is in the null space of $\partial_1\Phi\left(\frac{n-1}{\eta_0},0,\eta_0\right)$, then by the above equation $P_0[v]=0$ and $\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} v\,dz = 0$. The only such function is $v=0$, thus the null space is trivial. By considering $v=\bar{v}+\frac{\lambda\eta_0^{n-1}}{n\omega_n d(n-1)^{n-1}}$, for any $(\bar{v},\lambda)\in h_{e,0}^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)\times\mathbb{R}$, it follows that $\partial_1\Phi\left(\frac{n-1}{\eta_0},0,\eta_0\right) : h_e^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)\to h_{e,0}^{2,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)\times\mathbb{R}$ is bijective. We therefore use the implicit function theorem to obtain the function $\psi_{\eta_0} : U_{\eta_0}\to V_{\eta_0}$ with the property that for any $(u,\bar{u},\eta)\in V_{\eta_0}\times U_{\eta_0}$ we have

$$\Phi(u,\bar{u},\eta) = (0,0) \Leftrightarrow u = \psi_{\eta_0}(\bar{u},\eta).$$

$\square$

We note some additional properties of $\psi_{\eta 0}$. We have the representation

$$\psi_{\eta 0}(\bar{u}, \eta) = \bar{u} + \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)\, dz, \tag{7.13}$$

so $u = \bar{u} + C$, where C is some constant and, hence, $\frac{du}{dz} = \frac{d\bar{u}}{dz}$. The point $(0, \eta)$ corresponds to a cylindrical hypersurface of mean curvature η since

$$\psi_{\eta 0}(0, \eta) = \frac{n-1}{\eta}. \tag{7.14}$$

Lastly the following lemma gives the linearisations of $\psi_{\eta 0}$:

Lemma 7.2.2. *For any* $(\bar{u}, \eta) \in U_{\eta 0}$ *and* $\bar{v} \in h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$ *we have:*

$$\partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}] = \bar{v} - \frac{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1} \bar{v}\, dz}{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1}\, dz},$$

and

$$\partial_2 \psi_{\eta 0}(\bar{u}, \eta) = -\frac{(n-1)^n}{\eta^{n+1} \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1}\, dz}.$$

Proof. We start by taking the linearisation of the equation

$$\Phi\left(\psi_{\eta 0}(\bar{u}, \eta), \bar{u}, \eta\right) = (0, 0) \tag{7.15}$$

with respect to $\bar{u}$, using (7.12) we obtain:

$$\left(P_0\left[\partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}]\right] - \bar{v}, n\omega_n d \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1} \partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}]\, dz \right) = (0, 0).$$

Hence

$$\partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}] = \bar{v} + \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}]\, dz, \tag{7.16}$$

and

$$\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1} \partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}]\, dz = 0.$$

By substituting the first of these equations into the second we obtain

$$\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1} \bar{v}\, dz + \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta 0}(\bar{u}, \eta)^{n-1}\, dz \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \psi_{\eta 0}(\bar{u}, \eta)[\bar{v}]\, dz = 0.$$

This gives us

$$\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \psi_{\eta_0}(\bar{u},\eta)[\bar{v}]\, dz = -\frac{\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta_0}(\bar{u},\eta)^{n-1}\bar{v}\, dz}{\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta_0}(\bar{u},\eta)^{n-1}\, dz},$$

and combining with equation (7.16) gives the first result. To obtain the second result we take the derivative of (7.15), again using (7.12), with respect to η.

$$\left(P_0\left[\partial_2\psi_{\eta_0}(\bar{u},\eta)\right], n\omega_n d\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \psi_{\eta_0}(\bar{u},\eta)^{n-1}\partial_2\psi_{\eta_0}(\bar{u},\eta)\, dz + \frac{(n-1)^n}{\eta^{n+1}}\right)\right) = (0,0),$$

since the first component tells us that $\partial_2\psi_{\eta_0}(\bar{u},\eta)$ does not depend on z, the result is then obtained from the second component. $\square$

To simplify the notation we define, for $(\bar{u},\eta) \in U_{\eta_0}$:

$$\bar{F}_{\eta_0}(\bar{u},\eta) := H\left(\psi_{\eta_0}(\bar{u},\eta)\right) \tag{7.17}$$

and

$$\bar{G}_{\eta_0}(\bar{u},\eta) := P_0\left[\sqrt{1+\left(\frac{d\bar{u}}{dz}\right)^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{F}_{\eta_0}(\bar{u},\eta)\, d\bar{\mu}_{\eta_0}(\bar{r},\eta) - \bar{F}_{\eta_0}(\bar{u},\eta)\right)\right], \tag{7.18}$$

where $d\bar{\mu}_{\eta_0}(\bar{r},\eta) = \bar{\mu}_{\eta_0}(\bar{r},\eta)\, dz = \mu(\psi_0(\bar{r},\eta))\, dz$. We then obtain an equivalent flow to (7.9) (in a neighbourhood of η_0):

Lemma 7.2.3. *Let $\bar{u}(t)$ be a solution to the flow*

$$\frac{\partial \bar{u}}{\partial t} = \bar{G}_{\eta_0}(\bar{u},\eta),\ \bar{u}(0) = \bar{u}_0, \tag{7.19}$$

where $(u_0,\eta) \in U_{\eta_0}$. Then $\psi_{\eta_0}(\bar{u}(t),\eta)$ is a solution to (7.9). Conversely if $u(t)$, $t \in [0,\delta)$, is a solution to (7.9) such that $\left(P_0[u(t)], \frac{n-1}{\sqrt[n]{\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} u_0^n\, dz}}\right) \in U_{\eta_0}$, for each $t \in [0,\delta)$, then $P_0[u(t)]$ is a solution to (7.19) with $\eta = \frac{n-1}{\sqrt[n]{\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} u_0^n\, dz}}$.

Proof. We start by assuming $\bar{u}(t)$ is a solution to (7.19) and set $u(t) = \psi_{\eta_0}(\bar{u}(t),\eta)$.

We then use Lemma 7.2.2 to calculate the time derivative of $u(t)$ explicitly:

$$\begin{aligned}
\frac{\partial u}{\partial t} =& \partial_1 \psi_{\eta_0}(\bar{u}, \eta) \left[\frac{\partial \bar{u}}{\partial t}\right] \\
=& \partial_1 \psi_{\eta_0}(\bar{u}, \eta) \left[\bar{G}_{\eta_0}(\bar{u}(t), \eta)\right] \\
=& P_0 \left[G(u(t))\right] - \frac{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} P_0 \left[G(u(t))\right] \, dz}{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} \, dz} \\
=& G(u(t)) - \frac{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} G(u(t)) \, dz}{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} \, dz} \\
=& G(u(t)) - \frac{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} \sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2} \left(⨍_{\mathscr{S}^1_{\frac{d}{\pi}}} H(u(t)) \, d\mu_{u(t)} - H(u(t)) \right) dz}{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} \, dz} \\
=& G(u(t)) - \frac{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} \left(⨍_{\mathscr{S}^1_{\frac{d}{\pi}}} H(u(t)) \, d\mu_{u(t)} - H(u(t)) \right) d\mu_{u(t)}}{\int_{\mathscr{S}^1_{\frac{d}{\pi}}} u(t)^{n-1} \, dz} \\
=& G(u(t)).
\end{aligned}$$

The converse statement is obvious from the definition of $\bar{G}_{\eta_0}$. $\square$

In particular, this means that equations (7.9) and (7.19) have the same stationary solutions and that the curve $\left(0, \frac{n-1}{R}\right)$, for $R \in \mathbb{R}^+$ such that $(0, \frac{n-1}{R}) \in U_{\eta_0}$, is a family of stationary solutions to (7.19); we call this curve of solutions the trivial solution curve.

We seek to find nontrivial solution curves to the equation

$$\bar{G}_{\eta_0}(\bar{u}, \eta) = 0 \tag{7.20}$$

using bifurcation theory and hence find non-cylindrical CMC hypersurfaces. More precisely, we wish to prove that there exist non-cylindrical CMC hypersurfaces arbitrarily close to the cylinders of mean curvature $H_m := \frac{m\pi\sqrt{n-1}}{d}$.

Theorem 7.2.4. *The points $(0, H_m)$, for $m \in \mathbb{N}$, are the only bifurcation points on the trivial curve. That is, for each $m \in \mathbb{N}$ there exists a nontrivial continuously differentiable curve in $h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) \times \mathbb{R}^+$ through $(0, H_m)$:*

$$\{(\bar{r}_{m,s}, \eta_{m,s}) : s \in (-\delta, \delta), (\bar{r}_{m,0}, \eta_{m,0}) = (0, H_m)\} \subset U_{\eta_0}, \tag{7.21}$$

such that

$$\bar{G}_{H_m}\left(\bar{r}_{m,s}, \eta_{m,s}\right) = 0 \text{ for } s \in (-\delta, \delta),$$

and all solutions of $\bar{G}_{H_m}(\bar{u},\eta) = 0$ *in a neighbourhood of* $(0, H_m)$ *are either trivial solutions or on the nontrivial curve in (7.21).*

Proof. We start by determining the linearisation of $\bar{G}_{\eta_0}(\bar{u},\eta)$ with respect to the function variable at the point $(0,\eta)$. As it is clear that the projection P_0 commutes with the linearisation operator we will instead consider the functional:

$$\tilde{G}_{\eta_0}(\bar{u},\eta) := \sqrt{1+\left(\frac{d\bar{u}}{dz}\right)^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{F}_{\eta_0}(\bar{u},\eta)\,d\bar{\mu}_{\eta_0}(\bar{u},\eta) - \bar{F}_{\eta_0}(\bar{u},\eta)\right). \tag{7.22}$$

To simplify notation, we define $\bar{W}_{\eta_0}(\bar{u},\eta) = \ln(\bar{\mu}_{\eta_0}(\bar{u},\eta))$, so $d\bar{\mu}_{\eta_0}(\bar{u},\eta) = e^{\bar{W}_{\eta_0}(\bar{u},\eta)}\,dz$ and $\partial_1\, d\bar{\mu}_{\eta_0}(\bar{u},\eta)[\bar{v}] = \partial_1\bar{W}_{\eta_0}(\bar{u},\eta)[\bar{v}]\,d\bar{\mu}_{\eta_0}(\bar{u},\eta)$. We also use u' to represent $\frac{du}{dz}$ and drop the η_0 subscript. Note that $\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} g(\bar{u},\eta)\,d\bar{\mu}(0,\eta) = \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} g(\bar{u},\eta)\,dz$, where g is an arbitrary function. Taking the Fréchet derivative of (7.22) gives:

$$\begin{aligned}
\partial_1\tilde{G}(\bar{u},\eta)[\bar{v}] =& \frac{\bar{u}'\bar{v}'}{\sqrt{1+\bar{u}'^2}}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{F}(\bar{u},\eta)\,d\bar{\mu}(\bar{u},\eta) - \bar{F}(\bar{u},\eta)\right)\\
&+\sqrt{1+\bar{u}'^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1\bar{F}(\bar{u},\eta)[\bar{v}]\,d\bar{\mu}(\bar{u},\eta) - \partial_1\bar{F}(\bar{u},\eta)[\bar{v}]\right)\\
&+\sqrt{1+\bar{u}'^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{F}(\bar{u},\eta)\partial_1\bar{W}(\bar{u},\eta)[\bar{v}]\,d\bar{\mu}(\bar{u},\eta)\right.\\
&\qquad\left.-\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{F}(\bar{u},\eta)\,d\bar{\mu}(\bar{u},\eta)\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1\bar{W}(\bar{u},\eta)[\bar{v}]\,d\bar{\mu}(\bar{u},\eta)\right)\\
=&\frac{\bar{u}'\bar{v}'\tilde{G}(\bar{u},\eta)}{1+\bar{u}'^2} - \sqrt{1+\bar{u}'^2}\partial_1\bar{F}(\bar{u},\eta)[\bar{v}]\\
&+\sqrt{1+\bar{u}'^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1\bar{F}(\bar{u},\eta)[\bar{v}] - \frac{\tilde{G}(\bar{u},\eta)}{\sqrt{1+\bar{u}'^2}}\partial_1\bar{W}(\bar{u},\eta)[\bar{v}]\,d\bar{\mu}(\bar{u},\eta)\right).
\end{aligned} \tag{7.23}$$

From (7.17) we have:

$$\partial_1\bar{F}(\bar{u},\eta)[\bar{v}] = \partial H(\psi(\bar{u},\eta))\,[\partial_1\psi(\bar{u},\eta)[\bar{v}]]\,. \tag{7.24}$$

Using that $\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{v}\,dz = 0$ in Lemma 7.2.2 gives $\partial_1\psi(0,\eta)[\bar{v}] = \bar{v}$; so combining this with (7.23) and (7.24) gives:

$$\partial_1\tilde{G}(0,\eta)[\bar{v}] = \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial H\left(\frac{n-1}{\eta}\right)[\bar{v}]\;dz - \partial H\left(\frac{n-1}{\eta}\right)[\bar{v}]\,,$$

therefore $\partial_1 \tilde{G}(0,\eta)[\bar{v}]$ has zero mean for all $\bar{v}$, hence

$$\partial_1 \bar{G}(0,\eta)[\bar{v}] = \partial_1 \tilde{G}(0,\eta)[\bar{v}] = \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial H\left(\frac{n-1}{\eta}\right)[\bar{v}]\, dz - \partial H\left(\frac{n-1}{\eta}\right)[\bar{v}]\,.$$

Linearising (7.10) gives

$$\partial H(u)[v] = \frac{-v''}{(1+u'^2)^{3/2}} + \frac{3u''u'v'}{(1+u'^2)^{5/2}} - \frac{(n-1)v}{u^2\sqrt{1+u'^2}} - \frac{(n-1)u'v'}{u(1+u'^2)^{3/2}}, \tag{7.25}$$

hence

$$\begin{aligned}\partial_1 \bar{G}(0,\eta)[\bar{v}] &= \bar{v}'' + \frac{\eta^2}{n-1}\bar{v} - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{v}'' + \frac{\eta^2}{n-1}\bar{v}\, dz \\ &= \bar{v}'' + \frac{\eta^2}{n-1}\bar{v}, \end{aligned} \tag{7.26}$$

and

$$\partial^2_{12} \bar{G}(0,\eta)[\bar{v}] = \frac{2\eta}{n-1}\bar{v}. \tag{7.27}$$

The null space and range of (7.26) are easily calculated:

$$N\left(\partial_1 \bar{G}(0,\eta_0)\right) = \begin{cases} \operatorname{span}\left\{\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right\} & \eta_0 = H_m \text{ some } m \in \mathbb{N}, \\ \{0\} & \text{otherwise,}\end{cases}$$

$$\operatorname{Range}\left(\partial_1 \bar{G}(0,\eta_0)\right) = \begin{cases} h^{0,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) / \left\{\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right\} & \eta_0 = H_m \text{ some } m \in \mathbb{N}, \\ h^{0,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) & \text{otherwise.}\end{cases}$$

The implicit function theorem therefore guarantees bifurcation cannot occur on the trivial curve except at the points $(0, H_m)$, hence from now we consider just the points $(0, H_m)$; m can be thought of as a fixed natural number from here on. We set

$$\hat{v}_m = A_m \cos\left(\frac{H_m z}{\sqrt{n-1}}\right),$$

where $A_m := \left\|\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right\|^{-1}_{h^{2,\alpha}}$. We have

$$\partial^2_{12} \bar{G}(0,H_m)[\hat{v}_m] = \frac{2H_m A_m}{n-1}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right) \notin \operatorname{Range}\left(\partial_1 \bar{G}(0,H_m)\right), \tag{7.28}$$

therefore we can apply Theorem I.5.1 from [34] and conclude that bifurcation occurs at the point $(0, H_m)$ and we label the curve $(\bar{r}_{m,s}, \eta_{m,s})$. $\square$

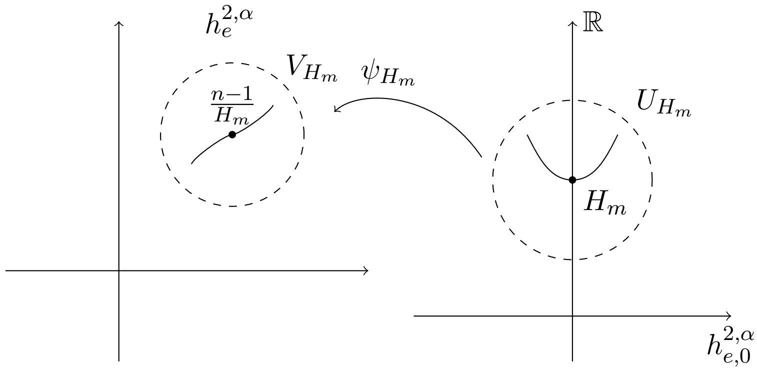


Figure 7.2: Non-trivial solution curve for equation (7.20) and its image under ψ_{H_m}

The curve (7.21) is shown in Figure 7.2, along with the curve $\psi_{H_m}(\bar{r}_{m,s}, \eta_{m,s})$, which is a curve of stationary solutions to (7.9).

Corollary 7.2.5. *There exists a continuously differentiable family of nontrivial axially symmetric CMC hypersurfaces that includes the cylinder of radius $\frac{n-1}{H_m}$, they are given by the profile curves $\rho_{m,s} := \psi_{H_m}(\bar{r}_{m,s}, \eta_{m,s})|_{[0,d]}$.*

In particular, this corollary states that any neighbourhood of a cylinder with mean curvature $H_1 = \frac{\pi\sqrt{n-1}}{d}$ contains CMC unduloids, which do not converge to a cylinder under the volume preserving mean curvature flow as they are stationary solutions. Therefore we obtain a counter example to Theorem 7.1.7 if $R = \frac{d\sqrt{n-1}}{\pi}$. In this way the theorem is sharp.

We now aim to study the stability of the nontrivial stationary solutions to (7.19) that are close to the bifurcation point $(0, H_1)$. We do this by investigating the shape of $\eta_{1,s}$. Note that for $m \geq 2$ the CMC unduloids are known to be unstable since the hypersurfaces contain a full period, [10].

Theorem 7.2.6. *The bifurcation curves in (7.21) satisfy:*

$$\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0} = 0 \tag{7.29}$$

and

$$\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} = \frac{(n^2 - 10n - 2)H_m^3 A_m^2}{12(n-1)^2}. \tag{7.30}$$

Proof. To calculate $\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0}$, we use equation (I.6.3) from [34]:

$$\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0} = \frac{-1}{2}\frac{\tilde{v}_m^*\left[\partial_{11}^2 \bar{G}_{H_m}(0, H_m)[\hat{v}_m, \hat{v}_m]\right]}{\tilde{v}_m^*\left[\partial_{12}^2 \bar{G}_{H_m}(0, H_m)[\hat{v}_m]\right]}, \tag{7.31}$$

where $\tilde{v}_m$ is an element not in the range of $\partial_1 \bar{G}_{H_m}(0, H_m)$ such that $\|\tilde{v}_m\|_{h^{0,\alpha}} = 1$, and $\tilde{v}_m^* \in h_{e,0}^{0,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)^*$, the dual space to the codomain, such that $\tilde{v}_m^*(\tilde{v}_m) = 1$ and $\tilde{v}_m^*(\partial_1 \bar{G}_{H_m}(0, H_m)[\bar{v}]) = 0$. Due to (7.28) we can take $\tilde{v}_m = B_m \cos\left(\frac{H_m z}{\sqrt{n-1}}\right)$, where $B_m := \left\|\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right\|^{-1}_{h^{0,\alpha}}$, and therefore

$$\tilde{v}_m^*[\bar{v}] = \frac{2}{B_m} \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{v} \cos\left(\frac{H_m z}{\sqrt{n-1}}\right) dz, \tag{7.32}$$

for all $\bar{v} \in h_{e,0}^{0,\alpha}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right)$.

Recalling (7.28) we have

$$\begin{aligned} \tilde{v}_m^*\left[\partial_{12}^2 \bar{G}_{H_m}(0, H_m)[\hat{v}_m]\right] =& \tilde{v}_m^*\left[\frac{2H_m A_m}{n-1} \cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right] \\ =& \frac{2H_m A_m}{(n-1)B_m}. \end{aligned} \tag{7.33}$$

Calculating $\partial_{11}^2 \bar{G}_{H_m}(0, H_m)[\hat{v}_m, \hat{v}_m]$ is a long process. We step through it gradually and obtain the formula in (7.43). We start by linearising (7.23) with respect to $\bar{u}$. Suppressing the H_m subscript, we calculate:

$$\begin{aligned} &\partial_{11}^2 \tilde{G}(\bar{u}, \eta)[\bar{v}, \bar{w}] \\ &= \frac{\bar{v}'\bar{w}'\tilde{G}(\bar{u}, \eta) + \bar{u}'\bar{v}'\partial_1 \tilde{G}(\bar{u}, \eta)[\bar{w}]}{1 + \bar{u}'^2} - \frac{2\bar{u}'^2 \bar{v}'\bar{w}'\tilde{G}(\bar{u}, \eta)}{(1 + \bar{u}'^2)^2} - \sqrt{1 + \bar{u}'^2}\,\partial_{11}^2 \bar{F}(\bar{u}, \eta)[\bar{v}, \bar{w}] \\ &\quad + \frac{\bar{u}'\bar{w}'}{\sqrt{1 + \bar{u}'^2}} \left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \bar{F}(\bar{u}, \eta)[\bar{v}] - \frac{\tilde{G}(\bar{u}, \eta)}{\sqrt{1 + \bar{u}'^2}} \partial_1 \bar{W}(\bar{u}, \eta)[\bar{v}]\, d\bar{\mu}(\bar{u}, \eta) - \partial_1 \bar{F}(\bar{u}, \eta)[\bar{v}]\right) \\ &\quad + \sqrt{1 + \bar{u}'^2} \left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_{11}^2 \bar{F}(\bar{u}, \eta)[\bar{v}, \bar{w}] - \frac{\partial_1 \tilde{G}(\bar{u}, \eta)[\bar{w}]\partial_1 \bar{W}(\bar{u}, \eta)[\bar{v}]}{\sqrt{1 + \bar{u}'^2}}\, d\bar{\mu}(\bar{u}, \eta)\right. \\ &\qquad - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \frac{\tilde{G}(\bar{u}, \eta)\partial_{11}^2 \bar{W}(\bar{u}, \eta)[\bar{v}, \bar{w}]}{\sqrt{1 + \bar{u}'^2}} - \frac{\bar{u}'\bar{w}'\tilde{G}(\bar{u}, \eta)\partial_1 \bar{W}(\bar{u}, \eta)[\bar{v}]}{(1 + \bar{u}'^2)^{3/2}}\, d\bar{\mu}(\bar{u}, \eta) \\ &\qquad + \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \left(\partial_1 \bar{F}(\bar{u}, \eta)[\bar{v}] - \frac{\tilde{G}(\bar{u}, \eta)\partial_1 \bar{W}(\bar{u}, \eta)[\bar{v}]}{\sqrt{1 + \bar{u}'^2}}\right) \partial_1 \bar{W}(\bar{u}, \eta)[\bar{w}]\, d\bar{\mu}(\bar{u}, \eta) \\ &\qquad - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \bar{F}(\bar{u}, \eta)[\bar{v}]\, d\bar{\mu}(\bar{u}, \eta) \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \bar{W}(\bar{u}, \eta)[\bar{w}]\, d\bar{\mu}(\bar{u}, \eta) \\ &\qquad \left. + \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \frac{\tilde{G}(\bar{u}, \eta)}{\sqrt{1 + \bar{u}'^2}} \partial_1 \bar{W}(\bar{u}, \eta)[\bar{v}]\, d\bar{\mu}(\bar{u}, \eta) \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1 \bar{W}(\bar{u}, \eta)[\bar{w}]\, d\bar{\mu}(\bar{u}, \eta)\right). \end{aligned}$$

Simplifying this by using (7.23) gives:

$$
\begin{aligned}
&\partial_{11}^2 \tilde{G}(\bar{u},\eta)[\bar{v},\bar{w}] \\
&\quad = \frac{\bar{u}'\left(\bar{v}'\partial_1\tilde{G}(\bar{u},\eta)[\bar{w}] + \bar{w}'\partial_1\tilde{G}(\bar{u},\eta)[\bar{v}]\right)}{1+\bar{u}'^2} + \frac{(1-2\bar{u}'^2)\bar{v}'\bar{w}'\tilde{G}(\bar{u},\eta)}{(1+\bar{u}'^2)^2} \\
&\qquad + \sqrt{1+\bar{u}'^2}\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_{11}^2\bar{F}(\bar{u},\eta)[\bar{v},\bar{w}] - \frac{\partial_1\tilde{G}(\bar{u},\eta)[\bar{w}]\partial_1\bar{W}(\bar{u},\eta)[\bar{v}]}{\sqrt{1+\bar{u}'^2}}\, d\bar{\mu}(\bar{u},\eta)\right. \\
&\qquad\qquad - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \frac{\partial_1\tilde{G}(\bar{u},\eta)[\bar{v}]\partial_1\bar{W}(\bar{u},\eta)[\bar{w}]}{\sqrt{1+\bar{u}'^2}} + \frac{\tilde{G}(\bar{u},\eta)\partial_{11}^2\bar{W}(\bar{u},\eta)[\bar{v},\bar{w}]}{\sqrt{1+\bar{u}'^2}}\, d\bar{\mu}(\bar{u},\eta) \\
&\qquad\qquad - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \frac{\tilde{G}(\bar{u},\eta)\partial_1\bar{W}(\bar{u},\eta)[\bar{v}]\partial_1\bar{W}(\bar{u},\eta)[\bar{w}]}{\sqrt{1+\bar{u}'^2}}\, d\bar{\mu}(\bar{u},\eta) - \partial_{11}^2\bar{F}(\bar{u},\eta)[\bar{v},\bar{w}] \\
&\qquad\qquad \left. + \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \frac{\bar{u}'\tilde{G}(\bar{u},\eta)\left(\bar{w}'\partial_1\bar{W}(\bar{u},\eta)[\bar{v}] + \bar{v}'\partial_1\bar{W}(\bar{u},\eta)[\bar{w}]\right)}{(1+\bar{u}'^2)^{3/2}}\, d\bar{\mu}(\bar{u},\eta)\right).
\end{aligned}
\tag{7.34}
$$

Evaluating at $(0, H_m)$ gives:

$$
\begin{aligned}
&\partial_{11}^2 \tilde{G}(0,H_m)[\bar{v},\bar{w}] \\
&\qquad = \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_{11}^2\bar{F}(0,H_m)[\bar{v},\bar{w}]\, dz - \partial_{11}^2\bar{F}(0,H_m)[\bar{v},\bar{w}] \\
&\qquad\quad - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_1\tilde{G}(0,H_m)[\bar{w}]\partial_1\bar{W}(0,H_m)[\bar{v}] + \partial_1\tilde{G}(0,H_m)[\bar{v}]\partial_1\bar{W}(0,H_m)[\bar{w}]\, dz.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\partial_{11}^2\bar{G}(0,H_m)[\bar{v},\bar{w}] =& P_0\left[\partial_{11}^2\tilde{G}(0,H_m)[\bar{v},\bar{w}]\right] \\
=& \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_{11}^2\bar{F}(0,H_m)[\bar{v},\bar{w}]\, dz - \partial_{11}^2\bar{F}(0,H_m)[\bar{v},\bar{w}].
\end{aligned}
\tag{7.35}
$$

We now linearise (7.24):

$$
\begin{aligned}
\partial_{11}^2\bar{F}(\bar{u},\eta)[\bar{v},\bar{w}] =& \partial^2 H\left(\psi(\bar{u},\eta)\right)\left[\partial_1\psi(\bar{u},\eta)[\bar{v}], \partial_1\psi(\bar{u},\eta)[\bar{w}]\right] \\
&+ \partial H\left(\psi(\bar{u},\eta)\right)\left[\partial_{11}^2\psi(\bar{u},\eta)[\bar{v},\bar{w}]\right],
\end{aligned}
\tag{7.36}
$$

evaluating at $(0, H_m)$ gives

$$
\partial_{11}^2\bar{F}(0,H_m)[\bar{v},\bar{w}] = \partial^2 H\left(\frac{n-1}{H_m}\right)[\bar{v},\bar{w}] + \partial H\left(\frac{n-1}{H_m}\right)\left[\partial_{11}^2\psi(0,H_m)[\bar{v},\bar{w}]\right]. \tag{7.37}
$$

To calculate $\partial^2_{11}\psi(0,H_m)[\bar v,\bar w]$ we use Lemma (7.2.2):

$$\partial^2_{11}\psi(\bar u,\eta)[\bar v,\bar w] = -\frac{(n-1)\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\psi(\bar u,\eta)^{n-2}\bar v\partial_1\psi(\bar u,\eta)[\bar w]\,dz}{\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\psi(\bar u,\eta)^{n-1}\,dz} + \frac{\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\psi(\bar u,\eta)^{n-1}\bar v\,dz\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}(n-1)\psi(\bar u,\eta)^{n-2}\partial_1\psi(\bar u,\eta)[\bar w]\,dz}{\left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\psi(\bar u,\eta)^{n-1}\,dz\right)^2}; \tag{7.38}$$

so using $\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar v\,dz = \fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar w\,dz = 0$, $\psi(0,H_m) = \frac{n-1}{H_m}$ and $\partial_1\psi_m(0,H_m)[\bar w] = \bar w$ we obtain

$$\partial^2_{11}\psi(0,H_m)[\bar v,\bar w] = -H_m\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar v\bar w\,dz. \tag{7.39}$$

Next we calculate the second variation of H from (7.25):

$$\begin{aligned}\partial^2 H(u)[v,w] =&\frac{3(u'v''w'+u'v'v''+u''v'w')}{(1+u'^2)^{5/2}} - \frac{15u''u'^2v'w'}{(1+u'^2)^{7/2}} + \frac{2(n-1)vw}{u^3\sqrt{1+u'^2}}\\ &+\frac{(n-1)(u'vw'+u'v'w)}{u^2(1+u'^2)^{3/2}} - \frac{(n-1)v'w'}{u(1+u'^2)^{3/2}} + \frac{3(n-1)u'^2v'w'}{u(1+u'^2)^{5/2}},\end{aligned} \tag{7.40}$$

hence

$$\partial^2 H\left(\frac{n-1}{H_m}\right)[v,w] = \frac{2H_m^3}{(n-1)^2}vw - H_m v'w'. \tag{7.41}$$

Substituting (7.37), (7.39) and (7.41) into (7.35) gives:

$$\begin{aligned}\partial^2_{11}\bar G(0,H_m)[\bar v,\bar w] =&\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\partial^2 H\left(\frac{n-1}{H_m}\right)[\bar v,\bar w] - H_m\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar v\bar w\,dz\partial H\left(\frac{n-1}{H_m}\right)[1]\;dz\\ &-\partial^2 H\left(\frac{n-1}{H_m}\right)[\bar v,\bar w] + H_m\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar v\bar w\,dz\partial H\left(\frac{n-1}{H_m}\right)[1]\\ =&\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\frac{2H_m^3}{(n-1)^2}\bar v\bar w - H_m\bar v'\bar w' + \frac{H_m^3}{n-1}\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar v\bar w\,dz\,dz\\ &-\frac{2H_m^3}{(n-1)^2}\bar v\bar w + H_m\bar v'\bar w' - \frac{H_m^3}{n-1}\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\bar v\bar w\,dz\\ =&H_m\bar v'\bar w' - \frac{2H_m^3}{(n-1)^2}\bar v\bar w - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}}H_m\bar v'\bar w' - \frac{2H_m^3}{(n-1)^2}\bar v\bar w\,dz,\end{aligned} \tag{7.42}$$

and consequently

$$
\begin{aligned}
&\partial_{11}^2 \bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m] \\
&\quad= \frac{H_m^3 A_m^2}{(n-1)} \sin^2\left(\frac{H_m z}{\sqrt{n-1}}\right) - \frac{2H_m^3 A_m^2}{(n-1)^2}\cos^2\left(\frac{H_m z}{\sqrt{n-1}}\right) \\
&\qquad - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \frac{H_m^3 A_m^2}{(n-1)} \sin^2\left(\frac{H_m z}{\sqrt{n-1}}\right) - \frac{2H_m^3 A_m^2}{(n-1)^2}\cos^2\left(\frac{H_m z}{\sqrt{n-1}}\right) dz \\
&\quad= \frac{H_m^3 A_m^2}{2(n-1)^2}\left((n-1)\left(1-\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right) - 2\left(1+\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right)\right. \\
&\qquad\qquad \left. - \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} (n-1)\left(1-\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right) - 2\left(1+\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right) dz\right) \\
&\quad= \frac{-(n+1)H_m^3 A_m^2}{2(n-1)^2}\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right).
\end{aligned} \tag{7.43}
$$

Therefore $\tilde{v}_m^*\left[\partial_{11}^2 \bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m]\right] = 0$ and hence $\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0} = 0$.

We will use equations (I.6.11) and (I.6.8) from [34] to calculate the second derivative

$$
\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} = \frac{-1}{3}\frac{\tilde{v}_m^*\left[\partial_{111}^3 \bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\right] + 3\tilde{v}_m^*\left[\partial_{11}^2 \bar{G}(0,H_m)[\hat{v}_m,\bar{w}_m]\right]}{\tilde{v}_m^*\left[\partial_{12}^2 \bar{G}(0,H_m)[\hat{v}_m]\right]}, \tag{7.44}
$$

where $\bar{w}_m$ solves

$$
\partial_{11}^2 \bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m] - \tilde{v}_m^*\left[\partial_{11}^2 \bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m]\right]\tilde{v}_m + \partial_1 \bar{G}(0,H_m)[\bar{w}_m] = 0. \tag{7.45}
$$

Using equations (7.43) and (7.26) we have that $\bar{w}_m$ satisfies

$$
\frac{-(n+1)H_m^3 A_m^2}{2(n-1)^2}\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right) + \bar{w}_m'' + \frac{H_m^2}{n-1}\bar{w}_m = 0,
$$

and hence

$$
\bar{w}_m = -\frac{(n+1)H_m A_m^2}{6(n-1)}\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right). \tag{7.46}
$$

Since $\tilde{v}_m^*[1] = 0$, we obtain from (7.42):

$$
\begin{aligned}
&\tilde{v}_m^*\left[\partial_{11}^2 \bar{G}(0,H_m)[\hat{v}_m,\bar{w}_m]\right] \\
&\qquad= \frac{(n+1)H_m^4 A_m^3}{6(n-1)^3}\tilde{v}_m^*\left[-2(n-1)\sin\left(\frac{2H_m z}{\sqrt{n-1}}\right)\sin\left(\frac{H_m z}{\sqrt{n-1}}\right)\right. \\
&\qquad\qquad \left. +2\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right] \\
&\qquad= \frac{(n+1)H_m^4 A_m^3}{6(n-1)^3}\tilde{v}_m^*\left[(n-1)\left(\cos\left(\frac{3H_m z}{\sqrt{n-1}}\right) - \cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right)\right. \\
&\qquad\qquad \left. +\cos\left(\frac{H_m z}{\sqrt{n-1}}\right) + \cos\left(\frac{3H_m z}{\sqrt{n-1}}\right)\right]
\end{aligned}
$$

$$\begin{aligned}
&\tilde{v}_m^* \left[\partial_{11}^2 \bar{G}(0, H_m)[\hat{v}_m, \bar{w}_m]\right] \\
&\qquad = \frac{(n+1)H_m^4 A_m^3}{6(n-1)^3} \tilde{v}_m^* \left[n \cos\left(\frac{3H_m z}{\sqrt{n-1}}\right) - (n-2)\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right] \\
&\qquad = \frac{-(n-2)(n+1)H_m^4 A_m^3}{6(n-1)^3 B_m}. \qquad (7.47)
\end{aligned}$$

Lastly we must calculate $\partial_{111}^3 \bar{G}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m]$, again this is a lengthy calculation and we do it in steps. We will first calculate $\partial_{111}^3 \tilde{G}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m]$, however even this is very complicated, so we will only calculate the important parts. In particular, we note that we will be setting $\bar{u} = 0$, so terms such as $\bar{u}' m(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \eta)$ will vanish, but more importantly any integral terms will vanish when acted on by the projection, P_0. Using (7.34) we find

$$\begin{aligned}
\partial_{111}^2 \tilde{G}(\bar{u}, \eta)[\bar{v}, \bar{w}, \bar{x}] =& \frac{\bar{x}' \left(\bar{v}' \partial_1 \tilde{G}(\bar{u}, \eta)[\bar{w}] + \bar{w}' \partial_1 \tilde{G}(\bar{u}, \eta)[\bar{v}]\right)}{1 + \bar{u}'^2} + \frac{\bar{v}' \bar{w}' \partial_1 \tilde{G}(\bar{u}, \eta)[\bar{x}]}{(1 + \bar{u}'^2)^2} \\
&- \sqrt{1 + \bar{u}'^2} \partial_{111}^3 \bar{F}(\bar{u}, \eta)[\bar{v}, \bar{w}, \bar{x}] + \bar{u}' m(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \eta) \\
&+ \sqrt{1 + \bar{u}'^2} \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} p(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \eta) \, d\bar{\mu}(\bar{u}, \eta),
\end{aligned}$$

for some operators $m(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \eta)$ and $p(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \eta)$. Since $\hat{v}_m$ is in the null space of $\partial_1 \tilde{G}(0, H_m)$ we have

$$\partial_{111}^2 \tilde{G}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m] = \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} p(0, \hat{v}_m, \hat{v}_m, \hat{v}_m, H_m) \, dz - \partial_{111}^3 \bar{F}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m].$$

Taking the projection gives

$$\begin{aligned}
\partial_{111}^2 \bar{G}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m] =& \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \partial_{111}^3 \bar{F}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m] \, dz \\
&- \partial_{111}^3 \bar{F}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m],
\end{aligned}$$

so that

$$\tilde{v}_m^* \left[\partial_{111}^2 \bar{G}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m]\right] = -\tilde{v}_m^* \left[\partial_{111}^3 \bar{F}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m]\right]. \qquad (7.48)$$

To calculate $\partial_{111}^3 \bar{F}(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m]$ we linearise (7.36):

$$\begin{aligned}
\partial_{111}^3 \bar{F}(\bar{u}, \eta)[\bar{v}, \bar{w}, \bar{x}] =& \partial^3 H\left(\psi(\bar{u}, \eta)\right)\left[\partial_1 \psi(\bar{u}, \eta)[\bar{v}], \partial_1 \psi(\bar{u}, \eta)[\bar{w}], \partial_1 \psi(\bar{u}, \eta)[\bar{x}]\right] \\
&+ \partial^2 H\left(\psi(\bar{u}, \eta)\right)\left[\partial_{11}^2 \psi(\bar{u}, \eta)[\bar{v}, \bar{x}], \partial_1 \psi(\bar{u}, \eta)[\bar{w}]\right] \\
&+ \partial^2 H\left(\psi(\bar{u}, \eta)\right)\left[\partial_{11}^2 \psi(\bar{u}, \eta)[\bar{w}, \bar{x}], \partial_1 \psi(\bar{u}, \eta)[\bar{v}]\right] \\
&+ \partial^2 H\left(\psi(\bar{u}, \eta)\right)\left[\partial_{11}^2 \psi(\bar{u}, \eta)[\bar{v}, \bar{w}], \partial_1 \psi(\bar{r}, \eta)[\bar{x}]\right] \\
&+ \partial H\left(\psi(\bar{u}, \eta)\right)\left[\partial_{111}^3 \psi(\bar{u}, \eta)[\bar{v}, \bar{w}, \bar{x}]\right].
\end{aligned}$$

Therefore, using (7.39),

$$\begin{aligned}\partial^3_{111}\bar{F}(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m] =&\partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\\ &-3H_m \fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \hat{v}_m^2\,dz\partial^2 H\left(\frac{n-1}{H_m}\right)[1,\hat{v}_m]\\ &+\partial H\left(\frac{n-1}{H_m}\right)\left[\partial^3_{111}\psi(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\right]. \end{aligned} \tag{7.49}$$

By considering (7.38) we see that $\partial^2_{11}\psi(\bar{u},\eta)[\bar{v},\bar{w}]$ maps into the constant functions, thus its linearisation does as well. This means that the final term in (7.49) will disappear when we act on it with the dual element, so we set $C_m := \partial^3_{111}\psi(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m]$. From $\fint_{\mathscr{S}^1_{\frac{d}{\pi}}}\cos^2\left(\frac{H_m z}{\sqrt{n-1}}\right)dz = \frac{1}{2}$ and equation (7.41) we obtain:

$$\begin{aligned}\partial^3_{111}\bar{F}(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m] =&\partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m,\hat{v}_m,\hat{v}_m]+\frac{H_m^2 C_m}{n-1}\\ &-\frac{3H_m A_m^2}{2}\left(\frac{2H_m^3 A_m}{(n-1)^2}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\right)\\ =&\partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m,\hat{v}_m,\hat{v}_m]+\frac{H_m^2 C_m}{n-1}\\ &-\frac{3H_m^4 A_m^3}{(n-1)^2}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right). \end{aligned} \tag{7.50}$$

Noting that any terms such as $u'm(u,v,w,x)$ or $u''p(u,v,w,x)$ will vanish, so we don't include them explicitly, we are able to easily linearise (7.40):

$$\begin{aligned}\partial^3 H(u)[v,w,x] =&\frac{3(v''w'x'+v'w''x'+v'w'x'')}{(1+u'^2)^{5/2}}-\frac{6(n-1)vwx}{u^4\sqrt{1+u'^2}}+\frac{(n-1)v'w'x}{u^2(1+u'^2)^{3/2}}\\ &+\frac{(n-1)(vw'x'+v'wx)}{u^2(1+u'^2)^{3/2}}+u'm(u,v,w,x)+u''p(u,v,w,x).\end{aligned}$$

Therefore

$$\begin{aligned}&\partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\\ &=9\hat{v}_m''\hat{v}_m'^2-\frac{6H_m^4\hat{v}_m^3}{(n-1)^3}+\frac{3H_m^2\hat{v}_m'^2\hat{v}_m}{n-1}\\ &=-\frac{6H_m^2\hat{v}_m'^2\hat{v}_m}{n-1}-\frac{6H_m^4\hat{v}_m^3}{(n-1)^3}\\ &=\frac{-6H_m^2 A_m}{n-1}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\left(\frac{H_m^2 A_m^2}{n-1}\sin^2\left(\frac{H_m z}{\sqrt{n-1}}\right)+\frac{H_m^2 A_m^2}{(n-1)^2}\cos^2\left(\frac{H_m z}{\sqrt{n-1}}\right)\right),\end{aligned}$$

where we used that, by definition, $\hat{v}_m$ is in the null space of (7.26). Simplifying we have

$$
\begin{aligned}
&\partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m,\hat{v}_m,\hat{v}_m] \\
&= \frac{-3H_m^4A_m^3}{(n-1)^3}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\left((n-1)\left(1-\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right)+1+\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right) \\
&= \frac{-3H_m^4A_m^3}{(n-1)^3}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)\left(n-(n-2)\cos\left(\frac{2H_m z}{\sqrt{n-1}}\right)\right) \\
&= \frac{-3H_m^4A_m^3}{2(n-1)^3}\left(2n\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)-(n-2)\left(\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)+\cos\left(\frac{3H_m z}{\sqrt{n-1}}\right)\right)\right) \\
&= \frac{-3H_m^4A_m^3}{2(n-1)^3}\left((n+2)\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)-(n-2)\cos\left(\frac{3H_m z}{\sqrt{n-1}}\right)\right). \qquad (7.51)
\end{aligned}
$$

Combining equations (7.51), (7.49) and (7.48) we arrive at

$$
\begin{aligned}
&\tilde{v}_m^*\left[\partial^2_{111}\bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\right] \\
&\qquad = \tilde{v}_m^*\left[\frac{3H_m^4A_m^3}{2(n-1)^3}\left((n+2)\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)-(n-2)\cos\left(\frac{3H_m z}{\sqrt{n-1}}\right)\right)\right. \\
&\qquad\qquad \left.+\frac{3H_m^4A_m^3}{(n-1)^2}\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)-\frac{H_m^2C_m}{n-1}\right] \\
&\qquad = \frac{9nH_m^4A_m^3}{2(n-1)^3B_m}. \qquad (7.52)
\end{aligned}
$$

Substituting (7.33), (7.47) and (7.52) into equation (7.44) gives:

$$
\begin{aligned}
\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} &= \frac{-(n-1)B_m}{6H_mA_m}\left(\frac{9nH_m^4A_m^3}{2(n-1)^3B_m}-\frac{(n-2)(n+1)H_m^4A_m^3}{2(n-1)^3B_m}\right) \\
&= \frac{(n^2-10n-2)H_m^3A_m^2}{12(n-1)^2}.
\end{aligned}
$$

□

We are now able to prove a surprising stability result for unduloids under the volume preserving mean curvature flow in high dimensions.

Corollary 7.2.7. *For $n \leq 10$ the unduloids close to the cylinder of radius $\frac{d\sqrt{n-1}}{\pi}$ are unstable equilibria of equation (1.2), while for $n \geq 11$ they are stable under volume preserving axially symmetric perturbations. That is, if $n \geq 11$ there exists $\epsilon > 0$ and a neighbourhood, $U_s \subset h^{2,\alpha}_{\frac{d}{dz}}([0,d])$, of $\rho_{1,s}$ for any $|s| \in (0,\epsilon)$, such that for any $\rho_0 \in U_s$ that encloses the same volume as $\rho_{1,s}$, the flow (1.6), with $M^n = \mathscr{C}^n_{R,d}$ and Neumann boundary condition, exists for all time and the solution $\rho(t)$ converges exponentially fast to $\rho_{1,s}$ as $t \to \infty$.*

Proof. We start by noting that the eigenvalues of $\partial_1 \bar{G}_{H_1}(0, H_1)$, except for the dominant one, lie in the open complex halfplane, $Re\,(\lambda) < 0$. Through a perturbation argument this is also true of the operator $\partial_1 \bar{G}_{H_1}(\bar{r}_{1,s}, \eta_{1,s})$ as long as s is small. We now determine the sign of the dominant eigenvalue of $\partial_1 \bar{G}_{H_1}(\bar{r}_{1,s}, \eta_{1,s})$ for s small. By Proposition I.7.2 in [34], there exists $\epsilon \in (0, \delta)$ and a continuously differentiable curve:

$$\{\lambda_{1,s} : |s| < \epsilon, \lambda_{1,0} = 0\} \subset \mathbb{R},$$

such that

$$\partial_1 \bar{G}_{H_1}(\bar{r}_{1,s}, \eta_{1,s})[\hat{v}_1 + v_{1,s}] = \lambda_{1,s}(\hat{v}_1 + v_{1,s}), \tag{7.53}$$

where $v_{1,s}$, for $|s| < \epsilon$, is a continuously differentiable curve in range of $\partial_1 \bar{G}_{H_1}(\bar{r}_{1,s}, \eta_{1,s})$ satisfying $v_{1,0} = 0$. Also, since $\frac{d\eta_{1,s}}{ds}|_{s=0} = 0$, we have that for $|s| < \epsilon$ (possibly making ϵ smaller),

$$\text{sign}(\lambda_{1,s}) = \text{sign}(H_1 - \eta_{1,s}), \tag{7.54}$$

by equation (I.7.46) in [34].

For $n \leq 10$ we see from equation (7.30) that $\eta_{1,s}$ has a local maximum at $\eta_{1,0} = H_1$ and hence the eigenvalue $\lambda_{1,s}$ is positive for $0 < |s| < \epsilon$. However, if $n \geq 11$, we see that $\eta_{1,s}$ has a local minimum at $\eta_{1,0} = H_1$ and hence the eigenvalue $\lambda_{1,s}$ is negative for $0 < |s| < \epsilon$. We also note that $\partial_1 \bar{G}_{H_1}(0, H_1)[\bar{v}]$ is the negative of an elliptic operator, so by Theorem 3.2.6 it is a sectorial operator on the little-Hölder spaces. The perturbation result in Proposition 3.2.8 then ensures that $\partial_1 \bar{G}_{H_1}(\bar{r}_{1,s}, \eta_{1,s})$ is sectorial for all $|s| < \epsilon$ (again possibly making ϵ smaller).

We can now apply Theorem 9.1.7 in [38] to obtain, in dimensions $2 \leq n \leq 10$, a nontrivial backward solution, $\bar{u}(t)$, of (7.19) with $\eta = \eta_{1,s}$ such that:

$$\|\bar{u}(t) - \bar{r}_{1,s}\|_{h^{2,\alpha}} \leq Ce^{\omega t}, \; t \leq 0, \tag{7.55}$$

where $C, \omega > 0$. By setting $\rho(t) := \psi_{H_1}\left(\bar{u}(t), \eta_{1,s}\right)|_{[0,d]}$ we obtain a nontrivial backward solution to (1.6) such that

$$\begin{aligned}
\|\rho(t) - \rho_{1,s}\|_{h^{2,\alpha}} &= \left\|\psi_{H_1}\left(\bar{u}(t), \eta_{1,s}\right)|_{[0,d]} - \psi_{H_1}\left(\bar{r}_{1,s}, \eta_{1,s}\right)|_{[0,d]}\right\|_{h^{2,\alpha}} \\
&\leq \left\|\psi_{H_1}\left(\bar{u}(t), \eta_{1,s}\right) - \psi_{H_1}\left(\bar{r}_{1,s}, \eta_{1,s}\right)\right\|_{h^{2,\alpha}} \\
&\leq b\left\|\bar{u}(t) - \bar{r}_{1,s}\right\|_{h^{2,\alpha}} \\
&\leq bCe^{\omega t}, \; t \leq 0,
\end{aligned}$$

where we have used that ψ_{H_1} is Lipschitz, with constant b. Thus the unduloid defined by $\rho_{1,s}$ is an unstable stationary solution of (1.2) when $2 \leq n \leq 10$.

When $n \geq 11$ we prove stability of the unduloid defined by $\rho_{1,s}$ by applying Theorem 9.1.7 in [38]. There exist $C, r, \omega > 0$ such that if $\|\bar{u}_0 - \bar{r}_{1,s}\|_{h^{2,\alpha}} < r$ then the solution, $u(t)$, of (7.19) with $\eta = \eta_{1,s}$ and initial condition $\bar{u}_0$ is defined for all $t \geq 0$ and satisfies

$$\|\bar{u}(t) - \bar{r}_{1,s}\|_{h^{2,\alpha}} + \left\|\bar{u}'(t)\right\|_{h^{0,\alpha}} \leq Ce^{-\omega t} \|\bar{u}_0 - \bar{r}_{1,s}\|_{h^{2,\alpha}}, \; t \geq 0. \tag{7.56}$$

This convergence is shown on the right hand side axes of 7.3. The function $\bar{r}_{1,s}$ is highlighted by a red dot and the equation (7.56) proves that any function on the red line converges to it under (7.19). Figure 7.3 also shows the mapping of this set under ψ_{H_1}, which gives all the functions, u, in a neighbourhood of $\psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})$ that satisfy $Vol(u) = Vol\left(\psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})\right)$.

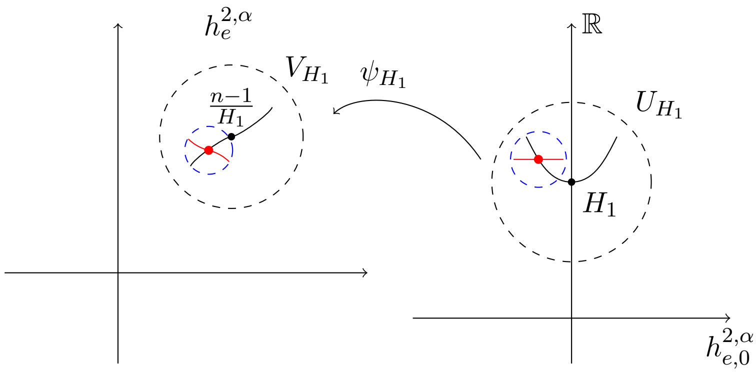


Figure 7.3: Sets of functions (red) that converge to nontrivial stationary solutions to the flows (7.9) (left) and (7.19) (right)

Considering ρ_0 such that $\|\rho_0 - \rho_{1,s}\|_{h^{2,\alpha}} < \frac{r}{4}$ and $Vol(\rho_0) = Vol(\rho_{1,s})$; then we have

$$\begin{aligned}
\|P_0[u_{\rho_0}] - \bar{r}_{1,s}\|_{h^{2,\alpha}} =& \|P_0\left[u_{\rho_0} - \psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})\right]\|_{h^{2,\alpha}} \\
\leq& 2\|u_{\rho_0} - \psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})\|_{h^{2,\alpha}} \\
\leq& 4\left\|\rho_0 - \psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})\,|_{[0,d]}\right\|_{h^{2,\alpha}} \\
<& r.
\end{aligned}$$

So, by the above calculations, there is a solution, $\bar{u}(t)$, of (7.19) with $\eta = \eta_{1,s}$ and $\bar{u}(0) = P_0[u_{\rho_0}]$ that satisfies (7.56). By setting $\rho(t) = \psi_{H_1}(\bar{u}(t), \eta_{1,s})\,|_{[0,d]}$ we obtain a solution to (1.6) with $\rho(0) = \psi_{H_1}(P_0[u_{\rho_0}], \eta_{1,s})\,|_{[0,d]} = u_{\rho_0}|_{[0,d]} = \rho_0$ such that

$$\begin{aligned}
\|\rho(t) - \rho_{1,s}\|_{h^{2,\alpha}} =& \left\|\psi_{H_1}(\bar{u}(t), \eta_{1,s})\,|_{[0,d]} - \psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})\,|_{[0,d]}\right\|_{h^{2,\alpha}} \\
\leq& \|\psi_{H_1}(\bar{u}(t), \eta_{1,s}) - \psi_{H_1}(\bar{r}_{1,s}, \eta_{1,s})\|_{h^{2,\alpha}} \\
\leq& b\,\|\bar{u}(t) - \bar{r}_{1,s}\|_{h^{2,\alpha}}
\end{aligned}$$

Therefore from (7.56):

$$\left\|\rho(t)-\rho_{1,s}\right\|_{h^{2,\alpha}} \leq bCe^{-\omega t}\left\|P_0[u_{\rho_0}]-\bar{r}_{1,s}\right\|_{h^{2,\alpha}},\ t\geq 0.$$

Thus the unduloid defined by $\rho_{1,s}$ is a stable stationary solution of (1.2) under volume preserving axially symmetric perturbations when $n \geq 11$. $\square$

7.3 Geometric Construction of Bifurcation Curves

In this section we consider an alternative method for constructing the bifurcation curves found in Section 7.2. We will use a representation of the axially symmetric CMC hypersurfaces to calculate the enclosed volume of such hypersurfaces and hence explicitly give a formula for $\eta_{1,s}$.

The n-dimensional axially symmetric CMC hypersurfaces were studied in [29], where the profile curve, $\rho(z)$, was shown to satisfy:

$$z=\int_{\rho(0)}^{\rho}\frac{1}{\sqrt{\left(\frac{x^{n-1}}{C_1+\frac{H}{n}x^n}\right)^2-1}}\,dx, \tag{7.57}$$

where C_1 is a constant and H is the mean curvature of the hypersurface. We note that for this representation the cylinders can only be treated through limits. Similarly, we can only treat the unduloids with half a period, i.e. when $m=1$. However, when we obtain the formula for the enclosed volume of $\rho_{1,s}$ we will be able to generalise it to any amount of periods. To obtain the bifurcation curve in Section 7.2 we apply the boundary conditions $\left.\frac{d\rho}{dz}\right|_{z=0}=\left.\frac{d\rho}{dz}\right|_{z=d}=0$ and we will also define $s:=\frac{\rho(d)-\rho(0)}{\rho(d)+\rho(0)}$.

The derivative, $\frac{d\rho}{dz}$, is given implicitly by

$$\frac{d\rho}{dz}=\sqrt{\left(\frac{\rho^{n-1}}{C_1+\frac{H}{n}\rho^n}\right)^2-1}.$$

From $\left.\frac{d\rho}{dz}\right|_{z=0}=0$ we obtain that $C_1=\rho(0)^{n-1}-\frac{H\rho(0)^n}{n}$ and hence

$$z=\int_{\rho(0)}^{\rho}\frac{1}{\sqrt{\left(\frac{x^{n-1}}{\rho(0)^{n-1}+\frac{H}{n}(x^n-\rho(0)^n)}\right)^2-1}}\,dx,$$

and using the change of variables $x=\rho(0)\bar{x}$ gives:

$$z=\rho(0)\int_{1}^{\frac{\rho}{\rho(0)}}\frac{1}{\sqrt{\left(\frac{\bar{x}^{n-1}}{1+\frac{H\rho(0)}{n}(\bar{x}^n-1)}\right)^2-1}}\,d\bar{x}, \tag{7.58}$$

with
$$\frac{d\rho}{dz} = \sqrt{\left(\frac{\left(\frac{\rho}{\rho(0)}\right)^{n-1}}{1+\frac{H\rho(0)}{n}\left(\left(\frac{\rho}{\rho(0)}\right)^{n}-1\right)}\right)^2 - 1}.$$

We next apply the boundary condition $\left.\frac{d\rho}{dz}\right|_{z=d} = 0$ and use the formula $\frac{\rho(d)}{\rho(0)} = \frac{1+s}{1-s}$ to obtain
$$H = \left(\frac{(1+s)^{n-1}-(1-s)^{n-1}}{(1+s)^n-(1-s)^n}\right)\frac{n(1-s)}{\rho(0)}, \tag{7.59}$$
thus:
$$z = \rho(0)\int_1^{\frac{\rho}{\rho(0)}} \frac{1}{\sqrt{\left(\frac{\bar{x}^{n-1}((1+s)^n-(1-s)^n)}{2s(1+s)^{n-1}+((1+s)^{n-1}-(1-s)^{n-1})(1-s)\bar{x}^n}\right)^2-1}}\,d\bar{x}. \tag{7.60}$$
Finally, evaluating at $z=d$ gives:
$$\rho(0) = d\left(\int_1^{\frac{1+s}{1-s}} \frac{1}{\sqrt{\left(\frac{\bar{x}^{n-1}((1+s)^n-(1-s)^n)}{2s(1+s)^{n-1}+((1+s)^{n-1}-(1-s)^{n-1})(1-s)\bar{x}^n}\right)^2-1}}\,d\bar{x}\right)^{-1}. \tag{7.61}$$

Equations (7.60) and (7.61) define the family of constant mean curvature hypersurfaces that meet the hyperplanes $z = 0, d$ orthogonally, i.e. $\rho_{1,s}$. We note here that as $s \to 0$ the values of $\rho_{1,s}(0)$ and $\rho_{1,s}(d)$ approach each other, so we should arrive at a cylinder. In fact, $\lim_{s\to 0}\rho_{1,s}(0) = \frac{d\sqrt{n-1}}{\pi}$ and so it is the cylinder with mean curvature H_1. Also the formula $(1-s)\rho_{1,s}(d) = (1+s)\rho_{1,s}(0)$ shows that as $s \to \pm 1$ one of the ends of the profile curve tends to the axis of rotation and the resulting axially symmetric hypersurface intersects the axis of rotation. In this case it represents a hemisphere. This can also be seen explicitly using (7.60) and (7.61):
$$\rho_{1,-1}(0) = d\left(\int_1^0 \frac{1}{\sqrt{\frac{1}{\bar{x}^2}-1}}\,d\bar{x}\right)^{-1} = d\left(\left[-\sqrt{1-\bar{x}^2}\right]_1^0\right)^{-1} = -d,$$
$$z = -d\int_1^{\frac{-\rho_{1,-1}}{d}} \frac{1}{\sqrt{\frac{1}{\bar{x}^2}-1}}\,dx = -d\left[-\sqrt{1-\bar{x}^2}\right]_1^{\frac{-\rho_{1,-1}}{d}} = d\sqrt{1-\left(\frac{\rho_{1,-1}}{d}\right)^2},$$
or equivalently $z^2+\rho_{1,-1}^2 = d^2$, a quarter circle of radius d centred at $(0,0)$.

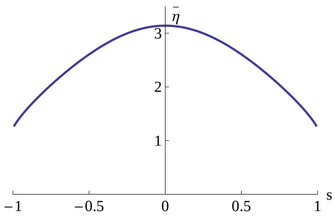
The n-dimensional shell method calculates the volume of a solid of revolution when integrating parallel to the axis of revolution:
$$\overline{Vol}(\rho_{1,s}) = S_{n-1}\int_{\rho_{1,s}(0)}^{\rho_{1,s}(d)} \rho^{n-1}z(\rho)\,d\rho, \tag{7.62}$$

where S_{n-1} is area of the unit $(n-1)$-sphere. In this situation $\overline{Vol}(\rho_{1,s})$ corresponds to the volume enclosed by the cylinder with length d and radius $\rho_{1,s}(d)$ and outside of the CMC hypersurface, therefore the volume enclosed by the CMC hypersurface is:

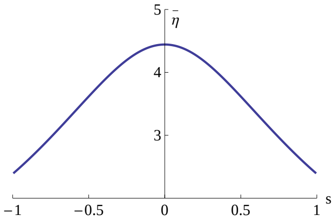
$$\begin{aligned} Vol(\rho_{1,s}) &= \omega_n \rho_{1,s}(d)^n d - n\omega_n \int_{\rho_{1,s}(0)}^{\rho_{1,s}(d)} \rho^{n-1} z(\rho)\, d\rho \\ &= \omega_n d \left(\left(\frac{1+s}{1-s}\right)^n \rho_{1,s}(0)^n \right. \\ &\qquad \left. - \frac{n\rho_{1,s}(0)^{n+1}}{d} \int_1^{\frac{1+s}{1-s}} \int_1^{\bar{y}} \frac{\bar{y}^{n-1}}{\sqrt{\left(\frac{\bar{x}^{n-1}\left(1+s-\frac{(1-s)^n}{(1+s)^{n-1}}\right)}{2s+\left(1-\left(\frac{1-s}{1+s}\right)^{n-1}\right)(1-s)\bar{x}^n} \right)^2 - 1}}\, d\bar{x}\, d\bar{y} \right), \end{aligned} \tag{7.63}$$

where we have used the change of variable $\rho = \rho_{1,s}(0)\bar{y}$ to get to the second line. In order to extend this to allow any number of periods we note that the volume of an unduloid made up of m half periods, will be m times the volume of a half period unduloid between plates a distance $\frac{d}{m}$ apart. Hence the volume of the m^{th} family of rotationally symmetric hypersurfaces is $Vol(\rho_{m,s}) = \frac{Vol(\rho_{1,s})}{m^n}$. Using equation (7.12) we have $\eta_{m,s} = (n-1)\sqrt[n]{\frac{\omega_n d}{Vol(\rho_{m,s})}} = m(n-1)\sqrt[n]{\frac{\omega_n d}{Vol(\rho_{1,s})}}$ and hence a parametrisation of the bifurcation parameter in (7.21) is obtained. The change of $\eta_{1,s}$ from being a maximum to a minimum can also be seen through plots of the normalised parameter $\bar{\eta}_{1,s} := \eta_{1,s} d$ for the different dimensions, see Figure 7.4.

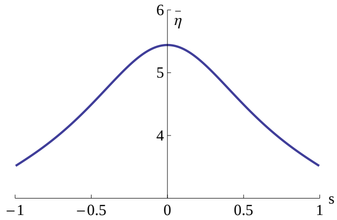
These plots confirm that the bifurcation parameter (volume enclosed) is a maximum (minimum) at the cylinder if $n \leq 10$, while for $n \geq 11$ it is a minimum (maximum) at the cylinder; see Figure 7.5 for a close up of the turning point for dimensions ten and eleven. Interesting phenomena are also apparent in dimensions eight and higher where additional turning points appear. In dimension eight, a local maximum and minimum of the enclosed volume occur within the family of unduloids. In dimensions nine and ten, the turning points separate from each other and these points are the global maximum and minimum volume of the family. In dimensions eleven and higher only the local minimum of the volume occurs and it remains a global minimum volume of the family. This behaviour is very intriguing and it would be of interest to know what is special about these unduloids.



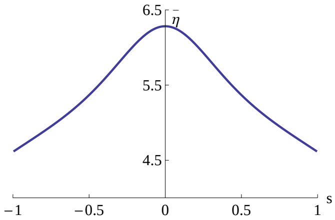
(a) $\bar{\eta}_{1,s}$ in dimension $n = 2$



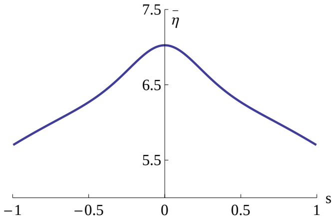
(b) $\bar{\eta}_{1,s}$ in dimension $n = 3$



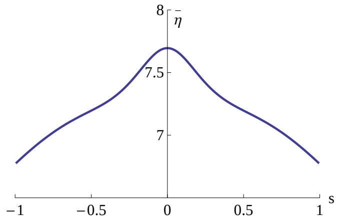
(c) $\bar{\eta}_{1,s}$ in dimension $n = 4$



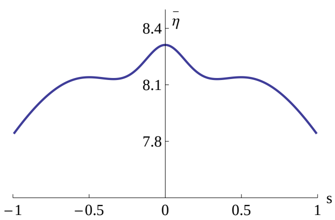
(d) $\bar{\eta}_{1,s}$ in dimension $n = 5$



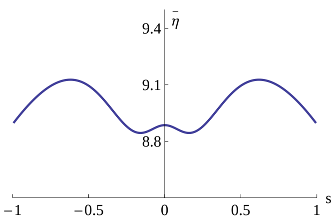
(e) $\bar{\eta}_{1,s}$ in dimension $n = 6$



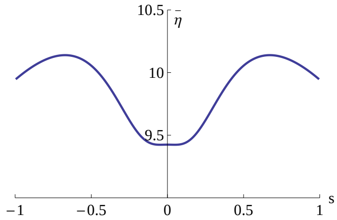
(f) $\bar{\eta}_{1,s}$ in dimension $n = 7$



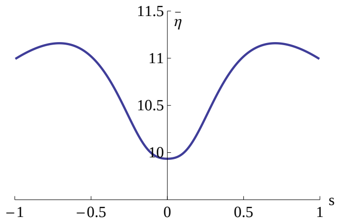
(g) $\bar{\eta}_{1,s}$ in dimension $n = 8$



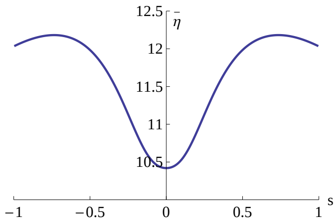
(h) $\bar{\eta}_{1,s}$ in dimension $n = 9$



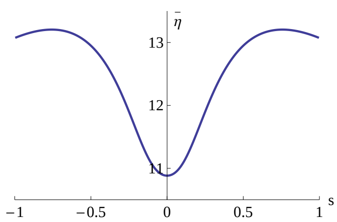
(i) $\bar{\eta}_{1,s}$ in dimension $n = 10$



(j) $\bar{\eta}_{1,s}$ in dimension $n = 11$

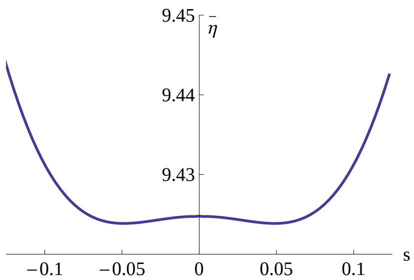


(k) $\bar{\eta}_{1,s}$ in dimension $n = 12$

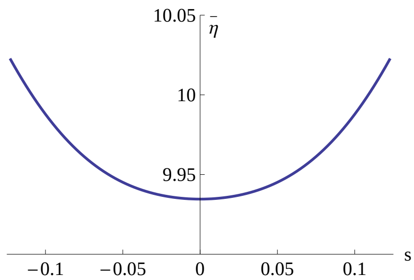


(l) $\bar{\eta}_{1,s}$ in dimension $n = 13$

Figure 7.4: Normalised bifurcation parameter in different dimensions



(a) Close up of $\bar{\eta}_{1,s}$ in dimension $n = 10$



(b) Close up of $\bar{\eta}_{1,s}$ in dimension $n = 11$

Figure 7.5: Turning point of the normalised bifurcation parameter for $n = 10, 11$

8

Mean Curvature Flow near Catenoids

In this chapter we consider the mean curvature flow equation in (1.1), whose stationary solutions are the minimal surfaces. We show how to analyse the stability of these minimal surfaces using the techniques in this thesis. We consider the case of the catenoid and find it is an unstable stationary solution to the flow, i.e. there are surfaces arbitrarily close to the catenoid that do not flow towards the catenoid. We will prove this by considering normal graphs over the catenoid:

$$\begin{aligned}\mathscr{C}\mathscr{A} := &\left\{\frac{1}{c}\left(\cosh\left(\frac{z-d}{c}\right)\cos(\theta), \cosh\left(\frac{z-d}{c}\right)\sin(\theta), \frac{z-d}{c}\right) \subset \mathbb{R}^3 :\right.\\ &\left.(\theta, z) \in [0, 2\pi) \times (0, d_1)\right\},\end{aligned} \tag{8.1}$$

where $d_1, c \in \mathbb{R}^+$ and $d \in \mathbb{R}$. The flow (1.1) is then equivalent to the evolution equation for the height function:

$$\frac{\partial \rho}{\partial t} = G_{ca}(\rho) := -\sqrt{1 + |\tilde{\nabla}\rho|^2} H(\rho), \quad \left.\frac{\partial \rho}{\partial z}\right|_{z=0,d_1} = 0. \tag{8.2}$$

Due to the presence of boundary conditions we work on the torus $\mathscr{T}^2_{d_1} := \mathbb{S}^1 \times \mathscr{S}^1_{\frac{d_1}{\pi}}$, with local coordinates (θ, z), and with the function spaces:

$$h^{k,\alpha}_e\left(\mathscr{T}^2_{d_1}\right) := \left\{u \in h^{k,\alpha}\left(\mathscr{T}^2_{d_1}\right) : u(\theta, z) = u(\theta, -z)\right\}.$$

Since $h^{k,\alpha}_e\left(\mathscr{T}^2_{d_1}\right)$ is a closed subspace of $h^{k,\alpha}\left(\mathscr{T}^2_{d_1}\right)$, for any $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, we can apply Lemma 3.1.2, where the projection operator is $u(\theta, z) \mapsto \frac{u(\theta,z)+u(\theta,-z)}{2}$, to conclude, using (3.10), that

$$\left(h^{0,\theta_1}_e\left(\mathscr{T}^2_{d_1}\right), h^{l,\theta_2}_e\left(\mathscr{T}^2_{d_1}\right)\right)_{\theta_0} = h^{\theta_0(l+\theta_2-\theta_1)+\theta_1}_e\left(\mathscr{T}^2_{d_1}\right), \tag{8.3}$$

for all $\theta_0, \theta_1, \theta_2 \in (0,1)$ and $l \in \mathbb{N}_0$ such that $\theta_0 (l + \theta_2 - \theta_1) + \theta_1 \notin \mathbb{N}$. By now defining $G_e(u)$, $u \in h_e^{k+2,\alpha}\left(\mathscr{T}_{d_1}^2\right)$, to be the even extension of $G_{ca}(u|_{\mathbb{S}^1 \times [0,d_1]})$, we have the equivalent PDE:

$$\frac{\partial u}{\partial t} = G_e(u). \tag{8.4}$$

It is of note that the instability result proved in this chapter could also be obtained by using that the catenoid is unstable as a critical point of the area functional, i.e. there are surfaces close to it with the same boundary but smaller area, [13, 37]. This means that the mean curvature flow starting from one of these surfaces cannot return to the catenoid, since the mean curvature flow decreases the area of a surface over time, [30].

We will use the following Theorems from [38] to determine the stability of catenoid:

Theorem 8.0.1 (Theorem 9.1.7 (ii) [38])**.** *Let $O \subset h_e^{2,\alpha}\left(\mathscr{T}_{d_1}^2\right)$ be a neighbourhood of 0 such that $G \in C^1\left(O, h_e^{0,\alpha}\left(\mathscr{T}_{d_1}^2\right)\right)$ is a nonlinear function with $G(0) = 0$ and $\partial G(0) = 0$. If $A : h_e^{2,\beta}\left(\mathscr{T}_{d_1}^2\right) \to h_e^{0,\beta}\left(\mathscr{T}_{d_1}^2\right)$, $\beta \in (0,\alpha)$, is sectorial and satisfies $\sigma_>(A) \neq \varnothing$ and $\omega_+ > 0$, see (1.17) and (1.19). Then the null solution of*

$$u'(t) = A[u(t)] + G(u(t)), \; u(0) = u_0 \tag{8.5}$$

is unstable in $h_e^{2,\alpha}\left(\mathscr{T}_{d_1}^2\right)$. Specifically, there exist nontrivial backward solutions to (8.5) converging to zero as t goes to negative infinity.

We let $P_>$ be the spectral projection associated with the spectral set $\sigma_>(A)$ and define $X^u := P_>\left(h_e^{0,\alpha}\left(\mathscr{T}_{d_1}^2\right)\right)$, $X^s := (I - P_>)\left(h_e^{2,\alpha}\left(\mathscr{T}_{d_1}^2\right)\right)$.

Theorem 8.0.2 (Theorem 9.1.8 [38])**.** *Let G and A satisfy the conditions in Theorem 8.0.1. If $\sigma(A) \cap i\mathbb{R} = \varnothing$, then for any $\alpha \in (\beta, 1)$ there exists:*

(i) $r_0, R_0 > 0$ and a Lipschitz continuous function

$$\phi : B_{X^u, r_0}(0) \to X^s,$$

differentiable at 0 with $\partial\phi(0) = 0$, such that for every u_0 belonging to the graph of ϕ problem (8.5) has a unique backward solution, $v(t)$, in $C\left((-\infty, 0], h_e^{2,\alpha}\left(\mathscr{T}_{d_1}^2\right)\right)$, such that $\|v\|_{L^\infty\left((-\infty,0], h_e^{2,\alpha}\right)} \leq R_0$. Moreover $e^{-\omega t} v(t) \in C\left((-\infty, 0], h_e^{2,\alpha}\left(\mathscr{T}_{d_1}^2\right)\right)$ for every $\omega \in (0, \omega_+)$. Conversely, if (8.5) has a backward solution v which satisfies the previous bound and $\|P_>[v(0)]\|_{h_e^{0,\alpha}} \leq r_0$ then $v(0) \in graph(\phi)$.

(ii) $r_1, R_1 > 0$ *and a Lipschitz continuous function*

$$\psi : B_{X^s, r_1}(0) \to X^u,$$

differentiable at 0 *with* $\partial\psi(0) = 0$, *such that for every* u_0 *belonging to the graph of* ψ *problem (8.5) has a unique solution,* $u(t)$, *in* $C\left([0,\infty), h_e^{2,\alpha}\left(\mathscr{T}^2_{d_1}\right)\right)$ *such that* $\|u\|_{L^\infty\left((0,\infty), h_e^{2,\alpha}\right)} \leq R_1$. *Moreover* $e^{\omega t}u(t) \in C\left([0,\infty), h_e^{2,\alpha}\left(\mathscr{T}^2_{d_1}\right)\right)$ *for every* $\omega \in (0, \omega_-)$. *Conversely, if (8.5) has a solution* u *which satisfies the previous bound and* $\|(I - P_>)[u(0)]\|_{h_e^{2,\alpha}} \leq r_1$ *then* $u(0) \in graph(\psi)$.

If in addition $G \in C^{k,1}\left(O, h_e^{0,\alpha}\left(\mathscr{T}^2_{d_1}\right)\right)$ *for* $k \in \mathbb{N}$, *then* ψ *and* ϕ *are* k *times differentiable, with Lipschitz* k*-th order derivatives.*

The graphs of ϕ and ψ are called the *local unstable manifold* and *local stable manifold* respectively.

Lemma 8.0.3. *For any* $v \in h_e^{2,\beta}\left(\mathscr{T}^2_{d_1}\right)$ *we have*

$$\partial G_e(0)[v] = \frac{1}{\cosh^2\left(\frac{|z|-d}{c}\right)}\left(\frac{1}{c^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{2v}{c^2\cosh^4\left(\frac{|z|-d}{c}\right)}.$$

Proof. We use that $\partial G_e(0)$ is the even extension of $\partial G_{ca}(0)$. From Lemma 4.1.2:

$$\begin{aligned}
\partial G_{ca}(0)[v] &= -\partial H(0)[v] \\
&= -\sum_{a=1}^{2} \partial\kappa_a(0)[v] \\
&= \sum_{a=1}^{2} \mathring{\zeta}^i_a \mathring{\zeta}^j_a \nabla_i \nabla_j v + \kappa_a(0)^2 v \\
&= \Delta_{\mathscr{C}\mathscr{A}} v + \frac{2v}{c^2\cosh^4\left(\frac{z-d}{c}\right)}
\end{aligned}$$

□

The linearised operator is therefore the negative of a uniformly elliptic operator, hence is sectorial in $h_e^{0,\beta}\left(\mathscr{T}^2_{d_1}\right)$ by Theorem 3.2.6, and we can use Theorem 4.3.1 to obtain existence for (8.4) and hence (8.2).

Theorem 8.0.4. *There exists* $\delta, r > 0$ *such that for any function* ρ_0 *satisfying the Neumann boundary conditions and* $\|\rho_0\|_{h^{2,\alpha}} \leq r$, *the equation (8.2) has a unique solution:*

$$\rho \in C\left([0,\delta), h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}\mathscr{A}}\right)\right) \cap C^1\left([0,\delta), h^{0,\alpha}\left(\overline{\mathscr{C}\mathscr{A}}\right)\right).$$

Moreover, the graph over the catenoid $\mathscr{CA}$, Ω_{ρ_0}, *has a mean curvature flow for* $t \in [0, \delta)$ *which is given, up to a tangential diffeomorphism, by* $\Omega_{\rho(t)}$.

We now consider the eigenvalues $\partial G_e(0)$.

Lemma 8.0.5. *The spectrum of* $\partial G_e(0) : h_e^{2,\beta}\left(\mathscr{T}_{d_1}^2\right) \to h_e^{0,\beta}\left(\mathscr{T}_{d_1}^2\right)$ *consists entirely of isolated eigenvalues with the first eigenvalue satisfying* $0 < \lambda_1 \leq \frac{2}{c^2}$. *Furthermore* $0 \notin \sigma(\partial G_e(0))$ *except in the exceptional case when* $d_1 = \tilde{d}$, *where* $\tilde{d}$ *is defined for* $d \in (-c\ln(1+\sqrt{2}), 0) \cup (c\ln(1+\sqrt{2}), \infty)$ *by the equations*

$$\frac{\tilde{d}}{c} - \frac{\cosh\left(\frac{\tilde{d}-d}{c}\right) + \cosh^3\left(\frac{\tilde{d}-d}{c}\right)}{\sinh\left(\frac{\tilde{d}-d}{c}\right)} = \frac{\cosh\left(\frac{d}{c}\right) + \cosh^3\left(\frac{d}{c}\right)}{\sinh\left(\frac{d}{c}\right)}, \ \tilde{d} > 0, \tag{8.6}$$

and undefined otherwise.

We note here that the function $f(z) = z - \frac{\cosh(z)+\cosh^3(z)}{\sinh(z)}$ has critical points at $z = \pm\ln\left(1+\sqrt{2}\right)$, with a local minimum at $z = -\ln\left(1+\sqrt{2}\right)$ and a local maximum at $z = \ln\left(1+\sqrt{2}\right)$, while being unbounded as $|z|$ tends to infinity or zero. Therefore for each $d \in (-c\ln(1+\sqrt{2}), 0) \cup (c\ln(1+\sqrt{2}), \infty)$ there is a single solution to the equations (8.6), while for other values of d there are no strictly positive solutions. Also note that if $d \in (-c\ln(1+\sqrt{2}), 0)$ then $\tilde{d} \in (d + c\ln(1+\sqrt{2}), \infty)$, while if $d \in (c\ln(1+\sqrt{2}), \infty)$ then $\tilde{d} \in (d - c\ln(1+\sqrt{2}), d)$.

Proof. We start by investigating the null space using separation of variables. Let $u(\theta, z) = X(\theta) Z(z)$ be an element of the null space. Therefore

$$\frac{1}{\cosh^2\left(\frac{|z|-d}{c}\right)}\left(\frac{1}{c^2}X''(\theta) Z(z) + X(\theta) Z''(z)\right) + \frac{2}{c^2\cosh^4\left(\frac{|z|-d}{c}\right)} X(\theta) Z(z) = 0,$$

or by rearranging

$$\frac{1}{X(\theta)} X''(\theta) + \left(\frac{c^2}{Z(z)} Z''(z) + \frac{2}{\cosh^2\left(\frac{|z|-d}{c}\right)}\right) = 0.$$

Both terms are therefore constant and we obtain $X''(\theta) = \xi X(\theta)$. Due to the periodic condition in the θ variable this is only possible for:

$$X_n(\theta) = C_1\cos(n\theta) + C_2\sin(n\theta),$$

where $n \in \mathbb{N}_0$. Therefore $Z(z)$ must satisfy

$$Z''(z) + \frac{1}{c^2}\left(\frac{2}{\cosh^2\left(\frac{|z|-d}{c}\right)} - n^2\right) Z(z) = 0.$$

The solutions to this equation are given in terms of the associated Legendre polynomials of the first and second kind, represented by P_m^n and Q_m^n respectively:

$$Z_n(z) = \bar{C}_1 P_1^n\left(\tanh\left(\frac{|z|-d}{c}\right)\right) + \bar{C}_2 Q_1^n\left(\tanh\left(\frac{|z|-d}{c}\right)\right). \tag{8.7}$$

P_1^n is given by

$$\begin{aligned} P_1^n(x) &= \frac{(-1)^n}{2}(1-x^2)^{n/2}\frac{d^{n+1}}{dx^{n+1}}\left(x^2+1\right) \\ &= \begin{cases} x & n=0, \\ -\sqrt{1-x^2} & n=1, \\ 0 & n\geq 2, \end{cases} \end{aligned}$$

and the first two associated Legendre polynomials of the second kind are

$$Q_1^0(x) = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1, \qquad Q_1^1(x) = \frac{(x^2-1)\log\left(\frac{1+x}{1-x}\right) - 2x}{2\sqrt{1-x^2}}.$$

For $n \geq 2$ $Q_1^n(x)$ has no zeros and a single turning point at $x=0$ for even n, and a single zero at $x=0$ and no turning points for odd n.

We now consider the three different cases, $n=0$, $n=1$ and $n\geq 2$, separately and enforce that $Z_n(z)$ has continuous first derivative at $z=0$ and $z=d$. For the $n=0$ case we have

$$Z_0(z) = \bar{C}_1 \tanh\left(\frac{|z|-d}{c}\right) + \bar{C}_2\left(\left(\frac{|z|-d}{c}\right)\tanh\left(\frac{|z|-d}{c}\right) - 1\right),$$

which has derivative

$$Z_0'(z) = \frac{|z|\operatorname{sech}^2\left(\frac{|z|-d}{c}\right)}{cz}\left(\bar{C}_1 + \bar{C}_2\left(\sinh\left(\frac{|z|-d}{c}\right)\cosh\left(\frac{|z|-d}{c}\right) + \left(\frac{|z|-d}{c}\right)\right)\right).$$

So to be continuously differentiable we require:

$$\bar{C}_1 + \left(0.5\sinh\left(\frac{-2d}{c}\right) - \frac{d}{c}\right)\bar{C}_2 = 0, \quad \bar{C}_1 + \left(0.5\sinh\left(\frac{2d_1-2d}{c}\right) + \frac{d_1-d}{c}\right)\bar{C}_2 = 0,$$

since $d_1 > 0$ and $0.5\sinh(2z)+z$ is a one-to-one function this system has only the trivial solution. Now we consider the $n=1$ case:

$$Z_1(z) = -\bar{C}_1\operatorname{sech}\left(\frac{|z|-d}{c}\right) - \bar{C}_2\left(\left(\frac{|z|-d}{c}\right)\operatorname{sech}\left(\frac{|z|-d}{c}\right) + \sinh\left(\frac{|z|-d}{c}\right)\right),$$

which has derivative

$$Z_1'(z) = \frac{|z|\operatorname{sech}\left(\frac{|z|-d}{c}\right)}{cz}\left(\bar{C}_1\tanh\left(\frac{|z|-d}{c}\right)\right.$$
$$\left.+\bar{C}_2\left(\left(\frac{|z|-d}{c}\right)\tanh\left(\frac{|z|-d}{c}\right)-1-\cosh^2\left(\frac{|z|-d}{c}\right)\right)\right).$$

Requiring continuous differentiability gives

$$\tanh\left(\frac{-d}{c}\right)\bar{C}_1+\left(\left(\frac{-d}{c}\right)\tanh\left(\frac{-d}{c}\right)-1-\cosh^2\left(\frac{-d}{c}\right)\right)\bar{C}_2=0,$$
$$\tanh\left(\frac{d_1-d}{c}\right)\bar{C}_1+\left(\left(\frac{d_1-d}{c}\right)\tanh\left(\frac{d_1-d}{c}\right)-1-\cosh^2\left(\frac{d_1-d}{c}\right)\right)\bar{C}_2=0.$$

Note that if either $d=0$ or $d_1=d$, then $\bar{C}_2=0$ and hence $\bar{C}_1=0$, so we only obtain the trivial solution. In the other cases we obtain

$$\bar{C}_1+\left(\frac{\cosh\left(\frac{d}{c}\right)+\cosh^3\left(\frac{d}{c}\right)}{\sinh\left(\frac{d}{c}\right)}-\frac{d}{c}\right)\bar{C}_2=0,$$
$$\bar{C}_1+\left(\frac{d_1-d}{c}-\frac{\cosh\left(\frac{d_1-d}{c}\right)+\cosh^3\left(\frac{d_1-d}{c}\right)}{\sinh\left(\frac{d_1-d}{c}\right)}\right)\bar{C}_2=0.$$

Therefore $\bar{C}_1=\bar{C}_2=0$ unless $d_1=\tilde{d}$, in which case:

$$Z_1(z)=\bar{C}_2\left(\left(\frac{\cosh\left(\frac{d}{c}\right)+\cosh^3\left(\frac{d}{c}\right)}{\sinh\left(\frac{d}{c}\right)}-\frac{|z|}{c}\right)\operatorname{sech}\left(\frac{|z|-d}{c}\right)-\sinh\left(\frac{|z|-d}{c}\right)\right).$$

Lastly we consider the $n\geq 2$ case:

$$Z_n(z)=\bar{C}_2Q_1^n\left(\tanh\left(\frac{|z|-d}{c}\right)\right).$$

The derivative is given by

$$Z_n'(z)=\frac{|z|\bar{C}_2}{cz}\operatorname{sech}^2\left(\frac{|z|-d}{c}\right)Q_1^{n\prime}\left(\tanh\left(\frac{|z|-d}{c}\right)\right),$$

so requiring differentiability gives

$$Q_1^{n\prime}\left(\tanh\left(\frac{-d}{c}\right)\right)=0,\qquad Q_1^{n\prime}\left(\tanh\left(\frac{d_1-d}{c}\right)\right)=0.$$

However $Q_1^{n\prime}(x)=0$ has at most one solution and since $\tanh(z)$ is one-to-one we find that only the trivial solution exists. Hence the operator has no null space except in the exceptional case of $d_1=\tilde{d}$, in which case the null space is the span of

$$u_1(\theta,z)=\left(\left(\frac{\cosh\left(\frac{d}{c}\right)+\cosh^3\left(\frac{d}{c}\right)}{\sinh\left(\frac{d}{c}\right)}-\frac{|z|}{c}\right)\operatorname{sech}\left(\frac{|z|-d}{c}\right)-\sinh\left(\frac{|z|-d}{c}\right)\right)\cos(\theta),$$

and

$$u_2(\theta, z) = \left(\left(\frac{\cosh\left(\frac{d}{c}\right) + \cosh^3\left(\frac{d}{c}\right)}{\sinh\left(\frac{d}{c}\right)} - \frac{|z|}{c}\right)\operatorname{sech}\left(\frac{|z|-d}{c}\right) - \sinh\left(\frac{|z|-d}{c}\right)\right)\sin(\theta).$$

To investigate the dominant eigenvalue we use that $\partial G_e(0)$ is self adjoint with respect to the inner product:

$$\langle u, v\rangle = \int_{\mathscr{S}^1_{\frac{d_1}{\pi}}}\int_{\mathbb{S}^1} uvc\cosh^2\left(\frac{|z|-d}{c}\right)\,d\theta\,dz,$$

and so has an associated bilinear form given by

$$\mathscr{L}(u,v) = \int_{\mathscr{S}^1_{\frac{d_1}{\pi}}}\int_{\mathbb{S}^1} \frac{1}{c}\frac{\partial u}{\partial\theta}\frac{\partial v}{\partial\theta} + c\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} - \frac{2uv}{c\cosh^2\left(\frac{|z|-d}{c}\right)}\,d\theta\,dz = -\left\langle \partial G_e(0)[u], v\right\rangle.$$

Therefore the largest eigenvalue is given by the Rayleigh quotient:

$$\lambda_1 = -\min\frac{\mathscr{L}(u,u)}{\langle u,u\rangle},$$

where we minimise over $u \in h_e^{1,\beta}\left(\mathscr{T}^2_{d_1}\right)$. We obtain an upper bound on the eigenvalue by ignoring the positive derivative terms in the integral and using that $\operatorname{sech}^2(z) \le \cosh^2(z)$ for all $z \in \mathbb{R}$:

$$\lambda_1 \le -\min\frac{-2\int_{\mathscr{S}^1_{\frac{d_1}{\pi}}}\int_{\mathbb{S}^1} u^2c^{-1}\operatorname{sech}^2\left(\frac{|z|-d}{c}\right)\,d\theta\,dz}{\int_{\mathscr{S}^1_{\frac{d_1}{\pi}}}\int_{\mathbb{S}^1} u^2c\cosh^2\left(\frac{|z|-d}{c}\right)\,d\theta\,dz} \le \frac{2}{c^2}.$$

To obtain the lower bound we calculate the Rayleigh quotient of $u = 1$:

$$\begin{aligned}
\lambda_1 &\ge -\frac{\mathscr{L}(1,1)}{\langle 1,1\rangle}\\
&= \frac{\int_{\mathscr{S}^1_{\frac{d_1}{\pi}}}\int_{\mathbb{S}^1} 2c^{-1}\operatorname{sech}^2\left(\frac{|z|-d}{c}\right)\,d\theta\,dz}{\int_{\mathscr{S}^1_{\frac{d_1}{\pi}}}\int_{\mathbb{S}^1} c\cosh^2\left(\frac{|z|-d}{c}\right)\,d\theta\,dz}\\
&= \frac{8\left(\tanh\left(\frac{d}{c}\right) - \tanh\left(\frac{d-d_1}{c}\right)\right)}{c\left(2d_1 + c\left(\sinh\left(\frac{2d}{c}\right) - \sinh\left(\frac{2d-2d_1}{c}\right)\right)\right)}\\
&> 0.
\end{aligned}$$

□

This allows us to apply Theorems 8.0.1 and 8.0.2 to (8.4) and, via its equivalence to (8.2), obtain the following result.

Theorem 8.0.6. *The finite catenoid is an unstable stationary solution to the mean curvature flow. That is there exists $r_0 > 0$ such that for any neighbourhood of zero, $O \subset h^{2,\alpha}_{\frac{\partial}{\partial z}}\left(\overline{\mathscr{C}\mathscr{A}}\right)$, there exists $\rho_0 \in O$ and $T > 0$ such that the solution to (8.2) satisfies $\|\rho(T)\|_{h^{2,\alpha}} > r_0$. Moreover if $d_1 \neq \tilde{d}$ then there exists local unstable and stable manifolds for the system. In particular there exists a $r_1 > 0$ such that if ρ_0 is an element of the stable manifold with $\rho_0 \in B_{h^{2,\alpha}_{\frac{\partial}{\partial z}}(\overline{\mathscr{C}\mathscr{A}}),r_1}(0)$ then the mean curvature flow of the surface defined by ρ_0 exists for all time and converges exponentially fast to a catenoid.*

Appendix A

Bifurcation Curves of Other Constant Mean Curvature Equations

In this section we return to studying the bifurcation of solutions to constant mean curvature equations, first covered in Section 7.2. We will consider an additional two constant mean curvature equations. The first such equation takes the same form as (7.20) however instead of the map $u = \psi_{\eta_0}(\bar{u}, \eta)$ we use $u = \bar{\psi}(\bar{u}, \eta) := \bar{u} + \frac{n-1}{\eta}$. Setting $\bar{F}_1(\bar{u}, \eta) = H\left(\bar{\psi}(\bar{u}, \eta)\right)$ and $d\bar{\mu}_1(\bar{u}, \eta) = \bar{\mu}_1(\bar{u}, \eta)\, dz = \mu\left(\bar{\psi}(\bar{u}, \eta)\right)\, dz$ we then have the equation:

$$\bar{G}_1(\bar{u}, \eta) := P_0 \left[\sqrt{1 + \bar{u}'(z)^2} \left(\fint_{\mathscr{S}^1_{\frac{d}{\pi}}} \bar{F}_1(\bar{u}, \eta)\, d\bar{\mu}_1(\bar{u}, \eta) - \bar{F}_1(\bar{u}, \eta) \right) \right] = 0, \tag{A.1}$$

note that now varying $\bar{u}$ does affect the volume of the hypersurface. The last equation we consider drops the global term and replaces it with the parameter η, i.e. it forces the corresponding hypersurface to have the same mean curvature as the cylinder it is a graph over, due to this we don't force our function to have zero mean.

$$\bar{G}_2(\bar{u}, \eta) := \eta - \bar{F}_1(\bar{u}, \eta) = 0, \tag{A.2}$$

note we have also left out the $\sqrt{1 + \bar{u}'^2}$ term as this equation no longer has relevance to a flow; however, it should be noted that this term does not affect the bifurcation properties.

Theorem A.0.1. *The points $(0, H_m)$, for $m \in \mathbb{N}$, are the only bifurcation points on the trivial curve of solutions to $\bar{G}_1(\bar{u}, \eta) = 0$. That is, for each $m \in \mathbb{N}$ there exists a nontrivial continuously differentiable curve in $h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) \times \mathbb{R}^+$ through $(0, H_m)$:*

$$\{(\bar{r}_{m,s}, \eta_{m,s}) : s \in (-\delta, \delta), (\bar{r}_{m,0}, \eta_{m,0}) = (0, H_m)\}\,, \tag{A.3}$$

such that

$$\bar{G}_1\left(\bar{r}_{m,s}, \eta_{m,s}\right) = 0 \text{ for } s \in (-\delta, \delta), \tag{A.4}$$

and all solutions of $\bar{G}_1\left(\bar{u}, \eta\right) = 0$ in a neighbourhood of $(0, H_m)$ are either trivial solutions or on the nontrivial curve in (A.3).

Furthermore

$$\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0} = 0, \tag{A.5}$$

and

$$\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} = \frac{H_m^3\left(n^2 - 4n - 8\right)}{12(n - 1 + H_m\sqrt{n-1} + H_m^2)^2}. \tag{A.6}$$

Proof. We start by noting that the function $\bar{\psi}$ is the first order (with respect to $\bar{u}$) approximation of ψ_{η_0} about the point $(0, \eta_0)$, so much of the analysis in the proofs of Theorems 7.2.4 and 7.2.6 can be used. In fact the only real change occurs when we calculate $\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0}$. In this case, instead of equation (7.50) we have

$$\partial^3_{111}\bar{F}_1(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m] = \partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m, \hat{v}_m, \hat{v}_m,] \tag{A.7}$$

and hence

$$\tilde{v}^*_m\left[\partial^2_{111}\bar{G}_1(0, H_m)[\hat{v}_m, \hat{v}_m, \hat{v}_m]\right] = \frac{3(n+2)H_m^4 A_m^3}{2(n-1)^3 B_m}. \tag{A.8}$$

So from (7.44) we obtain:

$$\begin{aligned}\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} &= \frac{-(n-1)B_m}{6H_m A_m}\left(\frac{3(n+2)H_m^4 A_m^3}{2(n-1)^3 B_m} - \frac{(n-2)(n+1)H_m^4 A_m^3}{2(n-1)^3 B_m}\right)\\ &= \frac{(n^2 - 4n - 8)H_m^3 A_m^2}{12(n-1)^2}.\end{aligned} \tag{A.9}$$

□

Corollary A.0.2. *For $2 \leq n \leq 5$ the bifurcation curves of equation (A.1) that pass through a trivial solution are subcritical, that is $\eta_{m,0}$ is a local maximum on the curve, while for $n \geq 6$ they are supercritical, that is $\eta_{m,0}$ is a local minimum on the curve.*

Theorem A.0.3. *The points $(0, H_m)$, for $m \in \mathbb{N}$, are the only bifurcation points on the trivial curve of solutions to $\bar{G}_2(\bar{u}, \eta) = 0$. That is, for each $m \in \mathbb{N}$ there exists a nontrivial continuously differentiable curve in $h^{2,\alpha}_{e,0}\left(\mathscr{S}^1_{\frac{d}{\pi}}\right) \times \mathbb{R}^+$ through $(0, H_m)$:*

$$\left\{(\bar{r}_{m,s}, \eta_{m,s}) : s \in (-\delta, \delta), (\bar{r}_{m,0}, \eta_{m,0}) = (0, H_m)\right\}, \tag{A.10}$$

such that

$$\bar{G}_2\left(\bar{r}_{m,s}, \eta_{m,s}\right) = 0 \text{ for } s \in (-\delta, \delta), \tag{A.11}$$

and all solutions of $\bar{G}_2(\bar{u}, \eta) = 0$ in a neighbourhood of $(0, H_m)$ are either trivial solutions or on the nontrivial curve in (A.10).

Furthermore

$$\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0} = 0, \tag{A.12}$$

and

$$\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} = \frac{H_m^3\left(n^2 - 10n - 10\right)}{12(n - 1 + H_m\sqrt{n-1} + H_m^2)^2}. \tag{A.13}$$

Proof. Due to the difference between this equation and the others it is easier to start from scratch and use known results as we proceed. Therefore, we linearise $\bar{G}_2$ with respect to the functional component:

$$\partial_1 \bar{G}_2(\bar{u}, \eta)[\bar{v}] = -\partial_1 \bar{F}_1(\bar{u}, \eta)[\bar{v}] = -\partial H\left(\bar{u} + \frac{n-1}{\eta}\right)[\bar{v}]. \tag{A.14}$$

Therefore, using equation (7.25), we have

$$\partial_1 \bar{G}_2(0, \eta)[\bar{v}] = \bar{v}'' + \frac{\eta^2}{n-1}\bar{v}, \ \partial^2_{12}\bar{G}_2(0, \eta)[\bar{v}] = \frac{2\eta}{n-1}\bar{v}. \tag{A.15}$$

These are the same operators found in the proof of Theorem 7.2.4. Hence, the same analysis gives the existence of bifurcation points on the trivial curve precisely at the points $(0, H_m)$.

Taking the second linearisation we obtain

$$\partial^2_{11}\bar{G}_2(\bar{u}, \eta)[\bar{v}, \bar{w}] = -\partial^2_{11}\bar{F}_1(\bar{u}, \eta)[\bar{v}, \bar{w}] = -\partial^2 H\left(\bar{u} + \frac{n-1}{\eta}\right)[\bar{v}, \bar{w}]. \tag{A.16}$$

So we use equation (7.41) to calculate:

$$\partial^2_{11}\bar{G}_2(0, H_m)[\bar{v}, \bar{w}] = \frac{-2H_m^3}{(n-1)^2}\bar{v}\bar{w} + H_m\bar{v}'\bar{w}', \tag{A.17}$$

and hence, since $\hat{v}_m$ has not changed, we have

$$\begin{aligned}\partial^2_{11}\bar{G}_2(0,H_m)[\hat{v}_m,\hat{v}_m] &= -\frac{2H_m^3}{(n-1)^2}\hat{v}_m^2 + H_m\hat{v}_m'^2 \\ &= H_m\left(\frac{-H_mA_m}{\sqrt{n-1}}\sin\left(\frac{H_mz}{\sqrt{n-1}}\right)\right)^2 - \frac{2H_m^3A_m^2}{(n-1)^2}\cos^2\left(\frac{H_mz}{\sqrt{n-1}}\right) \\ &= \frac{H_m^3A_m^2}{2(n-1)^2}\left((n-1)\left(1-\cos\left(\frac{2H_mz}{\sqrt{n-1}}\right)\right)\right. \\ &\qquad\qquad \left.-2\left(1+\cos\left(\frac{2H_mz}{\sqrt{n-1}}\right)\right)\right) \\ &= \frac{H_m^3A_m^2}{2(n-1)^2}\left(n-3-(n+1)\cos\left(\frac{2H_mz}{\sqrt{n-1}}\right)\right). \qquad (A.18)\end{aligned}$$

Therefore $\tilde{v}_m^*\left[\partial^2_{11}\bar{G}_2(0,H_m)[\hat{v}_m,\hat{v}_m]\right] = 0$ and hence $\left.\frac{d\eta_{m,s}}{ds}\right|_{s=0} = 0$.

To calculate $\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0}$ we first need to calculate $\bar{w}_m$. Substituting (A.18) and (A.15) into (7.45) gives

$$\frac{H_m^3A_m^2}{2(n-1)^2}\left(n-3-(n+1)\cos\left(\frac{2H_mz}{\sqrt{n-1}}\right)\right) + \bar{w}_m'' + \frac{H_m^2}{n-1}\bar{w}_m = 0, \qquad (A.19)$$

and hence

$$\bar{w}_m = -\frac{H_mA_m^2}{6(n-1)}\left(3(n-3)+(n+1)\cos\left(\frac{2H_mz}{\sqrt{n-1}}\right)\right). \qquad (A.20)$$

Therefore, using (A.17), we obtain

$$\begin{aligned}&\partial^2_{11}\bar{G}_2(0,H_m)[\hat{v}_m,\bar{w}_m] \qquad (A.21) \\ &= H_m\left(\frac{-H_mA_m}{\sqrt{n-1}}\sin\left(\frac{H_mz}{\sqrt{n-1}}\right)\right)\left(\frac{(n+1)H_m^2A_m^2}{3(n-1)^{3/2}}\sin\left(\frac{2H_mz}{\sqrt{n-1}}\right)\right) \\ &\quad + \frac{H_m^4A_m^3}{3(n-1)^3}\left(3(n-3)\cos\left(\frac{H_mz}{\sqrt{n-1}}\right) + (n+1)\cos\left(\frac{2H_mz}{\sqrt{n-1}}\right)\cos\left(\frac{H_mz}{\sqrt{n-1}}\right)\right) \\ &= \frac{H_m^4A_m^3(n+1)}{6(n-1)^3}\left(-(n-1)\left(\cos\left(\frac{H_mz}{\sqrt{n-1}}\right)-\cos\left(\frac{3H_mz}{\sqrt{n-1}}\right)\right)\right. \\ &\qquad\qquad \left.+\frac{6(n-3)}{n+1}\cos\left(\frac{H_mz}{\sqrt{n-1}}\right) + \left(\cos\left(\frac{H_mz}{\sqrt{n-1}}\right)+\cos\left(\frac{3H_mz}{\sqrt{n-1}}\right)\right)\right) \\ &= \frac{H_m^4A_m^3}{6(n-1)^3}\left(n(n+1)\cos\left(\frac{3H_mz}{\sqrt{n-1}}\right) - (n^2-7n+16)\cos\left(\frac{H_mz}{\sqrt{n-1}}\right)\right). \qquad (A.22)\end{aligned}$$

Thus

$$\tilde{v}_m^*\left[\partial^2_{11}\bar{G}_2(0,H_m)[\hat{v}_m,\bar{w}_m]\right] = \frac{-(n^2-7n+16)H_m^4A_m^3}{6(n-1)^3B_m}. \qquad (A.23)$$

Lastly we use (7.51) to calculate:

$$\begin{aligned}\partial^3_{111}\bar{G}(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m] =& -\partial^3 H\left(\frac{n-1}{H_m}\right)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\\ =&\frac{3H_m^4A_m^3(n+2)}{2(n-1)^3}\left(\cos\left(\frac{H_m z}{\sqrt{n-1}}\right)-\frac{n-2}{n+2}\cos\left(\frac{3H_m z}{\sqrt{n-1}}\right)\right),\end{aligned} \tag{A.24}$$

therefore, as in (A.8),

$$\tilde{v}_m^*\left[\partial^3_{111}\bar{G}_2(0,H_m)[\hat{v}_m,\hat{v}_m,\hat{v}_m]\right] = \frac{3(n+2)H_m^4A_m^3}{2(n-1)^3B_m}. \tag{A.25}$$

So by substituting this along with (A.23) into (7.44) we obtain

$$\begin{aligned}\left.\frac{d^2\eta_{m,s}}{ds^2}\right|_{s=0} &= \frac{-(n-1)B_m}{6H_mA_m}\left(\frac{3(n+2)H_m^4A_m^3}{2(n-1)^3B_m}-\frac{(n^2-7n+16)H_m^4A_m^3}{2(n-1)^3B_m}\right)\\ &= \frac{(n^2-10n+10)H_m^3A_m^2}{12(n-1)^2}.\end{aligned}$$

□

Corollary A.0.4. *For $2\le n\le 8$ the bifurcation curves of equation (A.2) that pass through a trivial solution are subcritical, while for $n\ge 9$ they are supercritical.*

Appendix B

Elementary Symmetric Function Identities

The aim of this appendix is to provide a complete proof of equation (2.6). This equation appears in [24] for the case where the hypersurface is convex; however here we will prove it for the elementary symmetric functions of an arbitrary matrix. We consider a matrix $A = \left(A^i_j\right)$, which has eigenvalues λ_a. The elementary symmetric functions are then given by

$$E_0 = 1,\ E_a = \sum_{1\leq b_1<\ldots<b_a\leq n} \prod_{i=1}^{a} \lambda_{b_i},\ 1 \leq a \leq n. \tag{B.1}$$

We first obtain a formula relating E_{a+1} to the previous elementary symmetric functions; this was proved in [41] but we reproduce the proof here for completeness.

Lemma B.0.1.

$$(a+1)E_{a+1} = \sum_{b=1}^{a+1} (-1)^{b+1} tr\left(A^b\right) E_{a+1-b},\ 0 \leq a \leq n-1. \tag{B.2}$$

Proof. We start by noting that the elementary symmetric functions are the coefficients of a certain polynomial:

$$\prod_{a=1}^{n} (1+\lambda_a t) = \sum_{a=0}^{n} E_a t^a.$$

Considering $|t| < \min_{1\leq a\leq n} |\lambda_a|^{-1}$ we can take the logarithm of both sides to remove the product:

$$\sum_{a=1}^{n} \ln|1+\lambda_a t| = \ln\left|\sum_{a=0}^{n} E_a t^a\right|,$$

and we are able to take the derivative with respect to t

$$\sum_{a=1}^{n} \lambda_a \left(1+\lambda_a t\right)^{-1} = \left(\sum_{a=1}^{n} a E_a t^{a-1}\right)\left(\sum_{a=0}^{n} E_a t^a\right)^{-1}.$$

Using the series expansion $(1+\lambda_a t)^{-1} = \sum_{b=0}^{\infty}(-1)^b \lambda_a^b t^b$, we obtain

$$\begin{aligned}
\sum_{a=1}^{n} a E_a t^{a-1} &= \left(\sum_{b=0}^{\infty}\sum_{a=1}^{n}(-1)^b \lambda_a^{b+1} t^b\right)\left(\sum_{a=0}^{n} E_a t^a\right)\\
&= \left(\sum_{b=0}^{\infty}(-1)^b \mathrm{tr}\left(A^{b+1}\right) t^b\right)\left(\sum_{a=0}^{n} E_a t^a\right)\\
&= \sum_{b=0}^{\infty}\sum_{a=0}^{n}(-1)^b \mathrm{tr}\left(A^{b+1}\right) E_a t^{a+b}.
\end{aligned}$$

Now we equate coefficients. Firstly for the coefficient of t^c where $c \geq n$ we obtain

$$0 = \sum_{b=c-n}^{c} (-1)^b \mathrm{tr}\left(A^{b+1}\right) E_{c-b},$$

while for the coefficient of t^c where $0 \leq c \leq n-1$ we obtain

$$(c+1)E_{c+1} = \sum_{b=0}^{c}(-1)^b \mathrm{tr}\left(A^{b+1}\right) E_{c-b},$$

which is the result. □

This lemma leads to a formula for the derivative of the elementary symmetric functions.

Proposition B.0.2.

$$\frac{\partial E_{a+1}}{\partial A^i_j} = \sum_{b=0}^{a}(-1)^b \left(A^b\right)^j_i E_{a-b},\ 0 \leq a \leq n-1. \tag{B.3}$$

Proof. The proof of this formula is by induction. We first show it is true when $a = 0$:

$$\frac{\partial E_1}{\partial A^i_j} = \frac{\partial \mathrm{tr}(A)}{\partial A^i_j} = \delta^j_i = \sum_{b=0}^{0}(-1)^b \left(A^b\right)^j_i E_{0-b}. \tag{B.4}$$

We now assume that (B.3) holds for all $0 \leq a \leq c-1$, where c is an integer between 1 and $n-1$. Taking the derivative of (B.2):

$$(c+1)\frac{\partial E_{c+1}}{\partial A^i_j} = \sum_{b=1}^{c+1}(-1)^{b+1} b \left(A^{b-1}\right)^j_i E_{c+1-b} + \sum_{b=1}^{c}(-1)^{b+1}\mathrm{tr}\left(A^b\right)\frac{\partial E_{c+1-b}}{\partial A^i_j},$$

and using (B.3) we obtain

$$
\begin{aligned}
(c+1)\frac{\partial E_{c+1}}{\partial A^i_j} =& \sum_{b=1}^{c+1}(-1)^{b+1}b\left(A^{b-1}\right)^j_i E_{c+1-b} \\
&+\sum_{b=1}^{c}(-1)^{b+1}\mathrm{tr}\left(A^b\right)\sum_{d=0}^{c-b}(-1)^d\left(A^d\right)^j_i E_{c-b-d} \\
=& \sum_{b=1}^{c+1}(-1)^{b+1}b\left(A^{b-1}\right)^j_i E_{c+1-b} \\
&+\sum_{d=0}^{c-1}(-1)^d\left(A^d\right)^j_i\sum_{b=1}^{c-d}(-1)^{b+1}\mathrm{tr}\left(A^b\right)E_{c-b-d} \\
=& \sum_{d=0}^{c}(-1)^d(d+1)\left(A^d\right)^j_i E_{c-d}+\sum_{d=0}^{c-1}(-1)^d\left(A^d\right)^j_i(c-d)E_{c-d} \\
=& \sum_{d=0}^{c}(-1)^d(c+1)\left(A^d\right)^j_i E_{c-d},
\end{aligned}
$$

where we used equation (B.2) to obtain the second last line. Cancelling the factor of $c+1$ gives that (B.3) is true for $a=c$. Hence by induction it is true of all $0\le a\le n-1$. $\square$

We now obtain the main result of the appendix, which is stated in terms of the Weingarten map in equation (2.6).

Corollary B.0.3.

$$
\frac{\partial E_{a+1}}{\partial A^i_j} = E_a\delta^j_i - A^j_k\frac{\partial E_a}{\partial A^i_k},\ \ 0\le a\le n-1. \tag{B.5}
$$

Proof. For $a=0$ the right hand side of (B.5) is δ^j_i so the equation follows from (B.4). For $1\le a\le n-1$ we calculate using equation (B.3):

$$
\begin{aligned}
\frac{\partial E_{a+1}}{\partial A^i_j}+A^j_k\frac{\partial E_a}{\partial A^i_k} =& \sum_{b=0}^{a}(-1)^b\left(A^b\right)^j_i E_{a-b}+A^j_k\sum_{b=0}^{a-1}(-1)^b\left(A^b\right)^k_i E_{a-1-b} \\
=& E_a\delta^j_i+\sum_{b=1}^{a}(-1)^b\left(A^b\right)^j_i E_{a-b}+\sum_{b=0}^{a-1}(-1)^b\left(A^{b+1}\right)^j_i E_{a-1-b} \\
=& E_a\delta^j_i+\sum_{b=1}^{a}(-1)^b\left(A^b\right)^j_i E_{a-b}+\sum_{b=1}^{a}(-1)^{b-1}\left(A^b\right)^j_i E_{a-b} \\
=& E_a\delta^j_i.
\end{aligned}
$$

$\square$

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