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by

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LEAD TIME DEMAND FOR SIMPLE EXPONENTIAL SMOOTHING

Abstract

A new simple formula is found to correct the underestimation of the standard deviation for total lead time demand when using simple exponential smoothing. The traditional formula for the standard deviation of lead time demand is to multiply the standard deviation for the one-period-ahead forecast error (estimated by using the residuals) by the square root of the number of periods in the lead time. It has been shown by others that the traditional formula significantly underestimates variation in the lead time demand when the mean of the process is somewhat changing and simple exponential smoothing is appropriate. This new formula allows one to see readily the significant size of the underestimation of the traditional formula and can easily be implemented in practice. The formula is derived by using a state space model for simple exponential smoothing.

KEYWORDS: Lead time demand, exponential smoothing, prediction intervals, safety stock

Introduction

The incorrect estimation of the variance for forecast error when simple exponential smoothing is used in computerized inventory control systems has been examined in many studies (for example see Johnston and Harrison[1], Newbold and Bos[2], and Harvey and Snyder[3]). Originally, R. G. Brown[4] proposed estimating the standard deviation for the total lead time demand by multiplying the standard deviation, σ, for the demand one-period-ahead by the square root of the number of periods, h, in the lead time (i.e., $\sigma\sqrt{h}$). We will call this estimate the "traditional formula." This formula is appropriate if a stationary model is generating the time series, that is, if the smoothing constant for the simple exponential smoothing process is zero. However, the point of using exponential smoothing is to account for the changing mean or level of the time series (a nonstationary process). Hence, it is important to determine the effect of a nonzero smoothing constant on the variance of the total lead time demand. Previous studies (Johnston and Harrison[1] and Harvey and Snyder[2]) have shown that this variance is significantly underestimated.

In this paper, we develop a simple formula for computing the variance of total lead time demand that accounts for both the length of the lead time and the size of the smoothing constant in simple exponential smoothing. For the generating model, we use one of the two models studied by Muth[5]. Both models assume constant variance for the error term. We choose the model with a single source of random error (SSRE) for several reasons. It is directly equivalent to the ARIMA(0,1,1) model with no restriction on the correlations. Muth's other model has two sources of random error and is appropriate only for time series for which the correlation of the differenced time series at lag 1 is negative. The SSRE model can be expressed in a form that allows one to see immediately the connection between the exponential smoothing method and the model (Ord, Koehler, and Snyder[6]). Most importantly, with the SSRE model one can easily compute the variance for the lead time demand and correct for the underestimation of the variance by the traditional formula. In fact, we show that the following adjustment factor replaces $\sqrt{h}$ in the traditional formula:

$$f(\alpha,h) = \sqrt{(h + \alpha(h-1)h(1 + \alpha(2h-1)/6))} \qquad (1)$$

where h is the length of the lead time and α is the smoothing constant.

In the next section we provide a background discussion of the models that assume constant variance for the one-period-ahead errors and that produce point forecasts that correspond to those from the simple exponential smoothing method. Then we use the SSRE model to compute the variance of the total lead time demand that contains the adjustment factor in Equation (1). We present a table that illustrates the importance of applying this adjustment factor in computing the standard deviation for different lead times and smoothing constants. We end the paper with a discussion of the practical implications of this adjustment factor for order-up-to levels in inventory control and for time series with trend.

Models for Simple Exponential Smoothing

The traditional formula ($\sigma\sqrt{h}$) for the standard deviation of total lead time demand (Brown[4]) is appropriate for the following model:

$$y_t = m + e_t \tag{2}$$

where y_t is the value of the time series at time t, m is the mean of this time series, σ^2 is its variance, and values of y_t at different time periods are independent of each other. If we have observed y_t for t = 1,2,...,n, then

Var(total lead time demand for h time periods)

$$= \mathrm{Var}\left(\sum_{j=1}^{h} y_{n+j}\right)$$

$$= \sigma^2 h \quad \text{(traditional formula)}$$

Clearly this model does not allow for the changing mean, m_t, that is implied by the method of simple exponential smoothing. Muth[5] proposed two models which he rationalized to underlie simple exponential smoothing. The first model has two sources of statistically independent random errors, e_t and v_t. The model has the form

$$y_t = m_{t-1} + e_t \tag{3a}$$

$$m_t = m_{t-1} + v_t \tag{3b}$$

where additional assumptions are that m_{t-1} is the mean of y_t and of m_t at time t-1, the e_t are independent, the v_t are independent, the variance of e_t is the constant σ_e^2, and the variance of

v_t is the constant σ_v^2.

The second model (SSRE) has one source of random error. We write it in a form that clearly shows the connection with simple exponential smoothing (Snyder[7]). This model is

$$y_t = m_{t-1} + e_t \tag{4a}$$

$$m_t = m_{t-1} + \alpha e_t \tag{4b}$$

where α is the smoothing parameter, m_{t-1} is the mean of y_t at time t-1, the e_t are independent, and σ^2 is the variance of e_t.

Both Models 3 and 4 are ARIMA(0,1,1) processes. However, for Model 3, it can be shown that

$$\mathrm{Cov}(\Delta y_t, \Delta y_{t-1}) = -\sigma_e^2/(\sigma_e^2 + \sigma_v^2)$$

which can never be positive. The SSRE model does not have this restriction and is equivalent to the ARIMA (0,1,1) model

$$y_t = y_{t-1} + e_t + (1 - \alpha)e_{t-1}. \tag{5}$$

Prediction

The prediction of the typical series value y_{n+j} beyond period n is conditional on the sample $y_1,y_2,\ldots,y_n$. For convenience, it is initially assumed that the seed level m_0, the smoothing parameter α, and the standard deviation σ are known exactly. Information until the end of period n is now denoted by $I_n = (y_1,y_2,\ldots,y_n,m_0,\alpha,\sigma)$. The problem is to find the distribution of $y_{n+j} \mid I_n$. For the SSRE Model 4, back-substitution of the recurrence relationship (4b) yields

$$m_{n+j} = m_n + \alpha \sum_{i=1}^{j} e_{n+i} \tag{6}$$

so that

$$y_{n+j} = m_n + \alpha \sum_{i=1}^{j-1} e_{n+i} + e_{n+j} \tag{7}$$

Thus, $E(y_{n+j} \mid I_n) = m_n$ as suggested above. Furthermore,

$$Var(y_{n+j} \mid I_n) = (j-1)\alpha^2 s^2 + s^2. \tag{8}$$

In inventory control applications, the primary interest is in total demand over a lead time h. By aggregating (7) we obtain

$$\sum_{j=1}^{h} y_{n+j} = hm_n + \sum_{j=1}^{h-1} (1 + (h-j)\alpha)e_{n+j} + e_{n+h} \tag{9}$$

Thus, the mean lead time demand, $E(\sum_{j=1}^{h} y_{n+j} \mid I_n) = hm_n$, is given by the usual formula.

The variance, however, is given by the more complex formula

$$Var(\sum_{j=1}^{h} y_{n+j} \mid I_n) = \sigma^2[h + \alpha(h-1)h(1 + \alpha(2h-1)/6)]. \tag{10}$$

$$= \sigma^2 f^2(\alpha,h)$$

where $f(\alpha,h)$ is defined in Equation 1. See the Appendix 1 for more details on the derivation.

When there is no structual change and, as a consequence $\alpha = 0$, the standard deviation reduces to the traditional square root formula $\sigma\sqrt{h}$. The second term under the square root symbol in Equation (1) (i.e., $\alpha(h-1)h(1+\alpha(2h-1)/6)$) may be viewed as the correction to the traditional formula required to allow for the impact of structural change. The significance of this correction term can be gauged from Table 1 where $f(\alpha,h)$ has been calculated for a range of values of the lead time and the smoothing parameter. Focussing, in particular, on the case of a lead time of h = 9 weeks, the first cell shows the $\sqrt{9}$, the factor that would be used in many conventional implementations of inventory control software. The remainder of the column shows much larger values than 3. This indicates that the standard deviation of total lead time demand is seriously underestimated if the traditional formula is used with forecasts from simple exponential smoothing in the usual context of positive smoothing parameter values. This highlights, in a more transparent way, the problem associated with the traditional formula originally exposed in Brown[4] and then elaborated by Johnston & Harrison[1] and by Harvey & Snyder[3].

- Insert Table 1 -

Conclusion

The adjustment factors are very important for computing safety stocks in inventory that are not too small. Order-up-to levels when using simple exponential smoothing would have the form:

$$hm_n + k\sigma f(\alpha,h)$$

where k would depend on the customer service objective and the type of distribution for e_t.

Since we have a model for exponential smoothing, we can use a maximum likelihood procedure to estimate m_0, a, and s. However, the replacement of $\sqrt{h}$ by $f(\alpha,h)$ can be implemented in any computerized system that currently uses simple exponential smoothing.

While Equation 1 is simple to program, it may not be immediately obvious how the standard deviation changes with h and a. A lower bound for $f(\alpha,h)$ is given by

$$g(\alpha,h) = \sqrt{h}\,(1 + \alpha(h-1)/2) \qquad (11)$$

where $g(\alpha,h)$ is correct for $\alpha = 0$ or $h = 1$ and an underestimate otherwise (see Appendix 2). When using $g(\alpha,h)$ as an approximation for $f(\alpha,h)$ in Table 1, the maximum error occurs for $\alpha = 1$ and $h = 10$ and is less than 12%. The limiting value of the error for all α as h increases is 13.4%. Those values contrast with the use of $\sqrt{h}$ in the traditional formula, where the corresponding errors are 84% and 100%.

From Equation 11 we can readily comprehend the impact of h and α on the standard deviation. Indeed the approximate adjustment $g(\alpha,h)$ will often be accurate enough and can be evaluated even more quickly than $f(\alpha,h)$.

It is also very easy to extend these results to the case when there is trend in the demand. Model 4 can be expanded to a model that underlies the Holt smoothing method as follows (Snyder[8]):

$$y_t = m_{t-1} + b_{t-1} + e_t$$

$$m_t = m_{t-1} + b_{t-1} + \alpha_1 e_t$$

$$b_t = b_{t-1} + \alpha_2 e_t$$

where m_t is the level of the time series at time t, b_t is the growth rate at time t, and α_1 and α_2 are parameters. If the level is changing ($\alpha_1 \neq 0$) and the growth rate is constant ($\alpha_2 = 0$), the adjustment factor is $f(\alpha_1,h)$. If the growth rate is changing, the adjustment factor must necessarily be larger and can be derived in the same manner as Equation 10.

Appendix 1

Derivation of the Variance for Total Lead Time Demand

Starting with Equation (9) for total lead time demand,

$$\mathrm{Var}\left(\sum_{j=1}^{h} y_{n+j} \,\middle|\, I_n\right) = \mathrm{Var}\left(hm_n + \sum_{j=1}^{h-1} (1 + (h-j)\alpha)e_{n+j} + e_{n+h} \,\middle|\, I_n\right)$$

$$= \sigma^2 \left(1 + \sum_{i=1}^{h-1} (1 + i\alpha)^2\right)$$

$$= \sigma^2 \left(1 + \sum_{i=1}^{h-1} (1 + 2i\alpha + i^2\alpha^2)\right)$$

$$= \sigma^2 \left(h + \sum_{i=1}^{h-1} 2i\alpha + \sum_{i=1}^{h-1} i^2\alpha^2\right)$$

$$= \sigma^2 \left[h + 2\alpha \frac{(h-1)(h)}{2} + \alpha^2 \frac{(h-1)(h)(2h-1)}{6}\right]$$

$$= \sigma^2 \left[h + \alpha (h-1)h(1 + \alpha (2h-1)/6)\right]$$

Appendix 2

Approximation to $f(\alpha,h)$

Completing the square for $1 + \alpha(h-1)$ in Equation 1, yields

$$f(\alpha,h) = \sqrt{h(1 + \alpha(h-1)/2)^2 + \alpha^2 h(h^2 - 1)/12}$$

An approximation to $f(\alpha,h)$ is given by

$$g(\alpha,h) = \sqrt{h}(1 + \alpha(h-1)/2),$$

which is clearly a lower bound.

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Table 1
Standard Deviation Adjustment Factor, f(α,h)

	Lead time, h									
α	1	2	3	4	5	6	7	8	9	10
0	1.00	1.41	1.73	2.00	2.24	2.45	2.65	2.83	3.00	3.16
0.1	1.00	1.49	1.91	2.31	2.70	3.09	3.48	3.87	4.27	4.67
0.2	1.00	1.56	2.10	2.64	3.19	3.77	4.36	4.98	5.62	6.28
0.3	1.00	1.64	2.29	2.98	3.70	4.47	5.27	6.12	7.00	7.92
0.4	1.00	1.72	2.49	3.32	4.22	5.18	6.19	7.27	8.39	9.57
0.5	1.00	1.80	2.69	3.67	4.74	5.89	7.12	8.43	9.80	11.24
0.6	1.00	1.89	2.90	4.03	5.27	6.62	8.06	9.59	11.21	12.91
0.7	1.00	1.97	3.11	4.39	5.81	7.35	9.00	10.76	12.62	14.58
0.8	1.00	2.06	3.32	4.75	6.34	8.07	9.94	11.93	14.04	16.26
0.9	1.00	2.15	3.53	5.11	6.88	8.81	10.89	13.11	15.46	17.94
1	1.00	2.24	3.74	5.48	7.42	9.54	11.83	14.28	16.88	19.62